$M \subseteq G$ Levi subgp, $\underline{G}$ splits over a tame extension

(σ, V_σ) a supercuspidal rep of M

Def A pair (K, ρ) consisting of a compact, open subgp $K \subset G$ an irred rep (ρ, V_ρ) of K is an $[M, \sigma]$-type if for all irred rep (π, V) of G the following are equivalent:

(i) $\pi \in \mathrm{Rep}(G)_{[M,\sigma]}$

(ii) $\rho \hookrightarrow \pi|_K$, i.e., $\mathrm{Hom}_K(\rho, \pi) \neq \{0\}$

restriction to K

Example: $G = SL_2(F)$, $T = M = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$

$(\mathcal{I}_\omega, \text{triv})$ is a $[T, \text{triv}]$-type

$\mathcal{I}_\omega = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix}_{\det = 1}$

Fact (Bushnell - Kutzko 1998)

$\tilde{K} \subset G$ compact-mod-center, open

$(\tilde{\rho}, V_{\tilde{\rho}})$ repr of $\tilde{K}$ s.th.

$\pi := \text{c-ind}_{\tilde{K}}^{G} \tilde{\rho}$ is irreducible, supercuspidal

Then $(\tilde{K}_{cpt}, \rho)$ is a $[G, \pi]$-type, where

$\tilde{K}_{cpt}$ = maximal compact subgp of $\tilde{K}$

ρ some irred rep in $\tilde{\rho}|\tilde{K}_{cpt}$

Thm (Bushnell - Kutzko, 1998)

If (K, ρ) is an $[M, \sigma]$-type,

then
$$\mathrm{Rep}(G)_{[M,\sigma]} \simeq \mathrm{Mod}\text{-}\mathcal{H}(G, K, \rho)$$
$$V \longmapsto V^{(K,\rho)}$$

$\mathcal{H}(G,K,\rho)$ $\parallel$ ← Hecke

$\mathcal{H}_G(V_\rho)$

$\{f : G \longrightarrow \mathrm{End}(V_\rho) \mid f(kgk') = \rho(k) f(g) \rho(k')$

$k, k' \in K, g \in G$ f cptly supp

Examples: $G = SL_2(F)$

a) $[M = G,\ \sigma = \operatorname{cind}_{\tilde{K}}^{G} \mathfrak{s}]$

$$\mathcal{H}(G, \underset{\underset{\tilde{K} = K}{\shortparallel}}{K_{cpt}}, \mathfrak{s}) \simeq \operatorname{End}_G(\operatorname{cind}_K^G \mathfrak{s}) \simeq \mathbb{C}$$

b) $[M = T,\ \mathrm{triv} = \sigma]$

$$\mathcal{H}(G, I_\omega, \mathrm{triv})$$

$$= \{ f : \underbrace{{}_{I_\omega}\backslash G / {}_{I_\omega}}_{N(T)/T_0 \,=:\, W_{aff}} \longrightarrow \mathbb{C},\ f \text{ cptly supp} \}$$

$$\simeq \langle s_0, s_1 \mid s_i^2 = 1 \rangle$$

$= \bigoplus_{\omega \in W_{aff}} \mathbb{C} \cdot T_\omega$ with relations generated by

- $T_\omega = T_{s_{i_1}} \cdot T_{s_{i_2}} \cdot \ldots \cdot T_{s_{i_n}}$ $\quad \omega = \underbrace{s_{i_1} s_{i_2} \cdot \ldots \cdot s_{i_n}}_{\text{shortest possible expression}}$
- $T_{s_i} \cdot T_{s_i} = q \cdot T_1 + (q-1) T_{s_i}$

$=: \mathcal{H}_{aff}(W_{aff}, q)$

Def K cpt, open subgp of G
K_M — " — M
(ρ, V_ρ) irred rep of K
(ρ_M, V_{ρ_M}) — " — K_M

The pair (K, ρ) is a <u>G-cover</u> of (K_M, ρ_M) if for every parabolic $P = M \cdot N \leq G$ and $\overline{P} = M \cdot \overline{N} \leq G$ with $P \cap \overline{P} = M$ we have:

(i) $K = (K \cap N)(K \cap M)(K \cap \overline{N})$
and $K \cap M = K_M$

(ii) $\rho|_{K_M} = \rho_M$, $\rho|_{K \cap N} = 1\!\!1_{V_\rho}$, $\rho|_{K \cap \overline{N}} = 1\!\!1_{V_\rho}$

(iii) For any irrep (π, V) of G, the restriction of
$$V \twoheadrightarrow V_N = V / \langle v - \pi(n)v \mid v \in V, n \in N \rangle$$
to $V^{(K,\rho)}$ is injective

$\uparrow$ subspace on which $\pi|_K$ acts via $\rho^{\oplus m}$

Example: $G = SL_2(F) \supset M = T$

(Iw, triv) is a G-cover of (T_0, triv)

<u>Thm</u> (Bushnell-Kutzko 1998)

Let (K_M, ρ_M) be an $[M, \sigma]$-type in M. Let (K, ρ) be a G-cover of (K_M, ρ_M).

Then (K, ρ) is an $[M, \sigma]$-type for G.

Construction of types ~~supercuspidal reps~~ à la Kim-Yu (+ twist by Fintzen-Kaletha-Spice)

Input: (i) $G^0 \subsetneq G^1 \subsetneq \dots \subsetneq G^{n-1} \subseteq G^n = G$ tame twisted Levi subgps ~~s.th. $Z(G^0)/Z(G)$ is anisotropic~~

(ii) $x \in \mathcal{B}(\underline{G}^0, F) \subset \mathcal{B}(\underline{G}^1, F) \subset \dots \subset \mathcal{B}(\underline{G}, F)$

$\cup \leftarrow$ "generic"

$x \in \mathcal{B}(\underline{M}^0, F)$ $\qquad M^0 \subset G^0$ Levi

s.th. x is a vertex in $\cancel{\mathcal{B}(\underline{G}^0, F)}$ $\mathcal{B}(M^0, F)$

(iii) $0 < r_0 < r_1 < \dots < r_{n-1}$

(iv) ϕ_i $(0 \leq i \leq n-1)$ a (G^{i+1}, G^i)-generic character of G^i of depth r_i

(v) ρ^0 an irred repr of $\cancel{G^0_x}$ $\underbrace{G^0_{x,0}(M^0_x)_{cpt}}_{K^0}$ such that $\rho^0|_{G^0_{x,0+}} = 1\!\!1_{V_{\rho^0}}$ and $\rho^0|_{G^0_{x,0}}$ is a cuspidal repr of $G^0_{x,0}/G^0_{x,0+} \simeq M^0_{x,0}/M^0_{x,0+}$

$\rightsquigarrow$ K $\cancel{\tilde{K}} = K^0$ $\cancel{G^0_x} \cdot G^1_{x, r_0/2} \cdot \ldots \cdot G^n_{x, r_{n-1}/2}$

ρ $\cancel{\tilde{\rho}} = \rho^0 \otimes \kappa$ $\leftarrow$ analogous to previous κ

$\kappa^{\mathrm{FKS}} \otimes \epsilon^{\mathrm{FKS}}$

__Thm__ (~~▮~~ Kim-Yu 2017, Fintzen 2021)

(K, ρ) is an $[M, \sigma]$-type.

If $p \nmid |\text{Weyl gp of } G|$, then $\forall [M, \sigma]$ $\exists (K, \rho)$ as above that is an $[M, \sigma]$-type.

Fix an input as above $\rightsquigarrow (K, \rho)$

__Fact__: $\operatorname{Supp} \mathcal{H}(G, K, \rho)$

$= K(\operatorname{Supp} \mathcal{H}(G^0, K^0, \rho^0))K$

__Prop__ (Adler - Fintzen - Mishra - Ohara, Aug 2024) "AFMO

$\exists$ subgp $N^\heartsuit \leq N_{G^0}(M^0, (M^0_x)_{cpt})$

such that ${}_{K^0}\backslash \operatorname{Supp}(\mathcal{H}(G^0, K^0, \rho^0))/_{K^0}$

$$\xleftarrow{\sim} \underbrace{N^\heartsuit / N^\heartsuit \cap (M^0_x)_{cpt}}_{\text{group}} =: W^\heartsuit$$

$\rightsquigarrow$ gp structure on Supp

<u>Thm</u> (AFMO, 08/2024)

$\exists$ a rep $\tilde{\kappa} : N^{\heartsuit}(K \cap M) \longrightarrow \mathrm{End}(V_\kappa)$
such that $\tilde{\kappa}|_{K\cap M} = \kappa$
and we have

$$\mathcal{I} : \mathcal{H}(G^\circ, K^\circ, \rho^\circ) \xrightarrow{\simeq} \mathcal{H}(G, K, \rho)$$

given by the following:
If $\varphi \in \mathcal{H}(G^\circ, K^\circ, \rho^\circ)$ is supported on $K^\circ n K^\circ$ with $n \in N^{\heartsuit}$,
then $\mathcal{I}(\varphi)$ is supported on KnK

and $\underset{\substack{\cap \\ \mathrm{End}(V_\rho) \\ \shortparallel \\ V_{\rho^\circ} \otimes V_\kappa}}{\mathcal{I}(\varphi)(n)} = d_n \cdot \underset{\substack{\cap \\ \mathrm{End}(V_{\rho^\circ})}}{\varphi(n)} \otimes \tilde{\kappa}(n)$

$$d_n = \sqrt{\frac{|K^\circ / nK^\circ n^{-1} \cap K^\circ|}{|K / nKn^{-1} \cap K|}}$$

$$\text{Cor: } \underset{\substack{\uparrow \\ (K,\rho) \text{ type}}}{\mathrm{Rep}(G)_{[M,\sigma]}} \simeq \mathrm{Rep}(G^\circ)_{[M^\circ, \underset{\substack{\uparrow \\ (K^\circ,\rho^\circ) \text{ depth-0}}}{\sigma^\circ}]}$$

Thm (AFMO, 08/2024)

Morris '93

$$W^{\heartsuit} \simeq \underbrace{W(\mathfrak{s})_{\text{aff}}}_{\text{affine Weyl gp}} \rtimes \Omega(\mathfrak{s})$$

$$\mathcal{H}(G, K, \mathfrak{s}) \simeq \mathcal{H}(W(\mathfrak{s})_{\text{aff}}, \{q_s\}) \rtimes \mathbb{C}[\Omega(\mathfrak{s}), \mu]$$

some 2-cocycle

$\Omega(\mathfrak{s}) \times \Omega(\mathfrak{s}) \to \mathbb{C}^{\times}$

$$\begin{array}{ccc} M^\circ & \subset & G^\circ \\ \cap & & \cap \\ M & \subset & G \\ \overset{?}{\|} & & \end{array}$$

$$M := \mathrm{Cent}_G(\mathrm{Zsplit}(M^\circ))$$