<u>Thm</u> (Moy - Prasad 1994 / 1996)

Let $x \in \mathcal{B}(\underline{G}, F)$ be a vertex.

$G_x := \mathrm{Stab}_G(x) = \{ g \in G \mid g \cdot x = x \}$

Let (ρ, V_ρ) be an irred rep of G_x

s.th. (i) $\rho|_{G_{x,0+}} = \mathbb{1}_{V_\rho}$

(ii) $\rho|_{G_{x,0}}$ is a cuspidal rep of $G_{x,0}/G_{x,0+}$

Then $\text{c-ind}_{G_x}^{G} V_\rho$ is an irred supercuspidal repr of depth 0.

All depth-zero supercuspidal reps arise in this way.

Recall: $\text{c-ind}_K^G V_\rho$

$$= \left\{ f : G \longrightarrow V_\rho \;\middle|\; \begin{array}{l} f(kg) = \rho(k) f(g) \;\; \forall k \in K, g \in G \\ f \text{ compactly supp. mod } K \end{array} \right\}$$

Positive depth reps

Assume from now on that G splits over a tamely ramified extension

<u>Def</u>: We call $\underline{G}' \subset \underline{G}$ a (tame) <u>twisted Levi subgroup</u> if $\exists$ a (tamely ramified) field extension E/F such that

$$\underline{G}' \times E \subseteq G \underset{F}{\times} E \text{ is a Levi}$$

subgroup (of a parab.) of $\underline{G}$.

Example: $G = SL_2(\mathbb{Q}_p)$ $p \neq 2$

$$T_{an} := G' = \left\{ \begin{pmatrix} a & b \\ pb & a \end{pmatrix} \in SL_2(\mathbb{Q}_p) \right\}$$

over $E = \mathbb{Q}_p(\sqrt{p})$:

$$\underline{G}'(E) = \left\{ \begin{pmatrix} a & b \\ pb & a \end{pmatrix} \in SL_2(\mathbb{Q}_p(\sqrt{p})) \right\}$$

$$\underset{\text{conjugate}}{\sim} \left\{ \begin{pmatrix} a + b\sqrt{p} & 0 \\ 0 & a - b\sqrt{p} \end{pmatrix} \in SL_2(\mathbb{Q}_p(\sqrt{p})) \right\}$$

$$= \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}$$

<u>Def</u>: Assume $p \nmid |\text{Weyl gp of } G|$
Let $G' \subset G$ be a twisted Levi subgps. $T' \subset G'$ a tame maximal torus, i.e., $\underline{T}' \times E \subset \underline{G}' \times E$ is a split maximal torus for some E/F tamely ramified ext.
A character ϕ of G' is called <u>(G, G')-generic of depth r</u> if
ϕ is of depth r and
$\phi(Nm_{E/F}(\check{\alpha}((E^\times)_r))) \neq 1$
norm E/F
$\forall \alpha \in \underline{\Phi(G_E, T_E) - \Phi(G'_E, T'_E)}$

e.g. $G = GL_2(\mathbb{Q}_p)$

$\boxed{(F^\times)_r := 1 + \varpi^r \mathcal{O} \quad r > 0}$

$G' = \left\{\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right\} = T'$

$\Phi(G, T') - \underbrace{\Phi(G', T')}_{\{\}} = \{\pm\alpha\}$

$\alpha : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto t_1 t_2^{-1} \qquad \check{\alpha} : t \longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$

example: $G = GL_2(\mathbb{Q}_p)$ $\quad p = 7$

$G' = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right\} = T'$

Let $\psi : \mathbb{Q}_7^\times \longrightarrow \mathbb{C}^\times$ s.th.

$\psi : 1 + 7 \cdot \mathbb{Z}_7 \longrightarrow \mathbb{C}^\times$ nontrivial

$(F^\times)_{1+} = 1 + 7^2 \cdot \mathbb{Z}_7 \overset{\psi}{\longmapsto} 1$

$\rightsquigarrow$ depth of ψ is 1

$\phi_1 : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto \psi(t_1)$

$\phi_2 : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto \psi(t_2)$

$\phi_3 : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto \psi(t_1 t_2^{-1})$

generic of depth 1

$\phi_4 : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto \psi(t_1 t_2)$ not generic

cf. $\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ is regular semisimple

$\iff t_1 \neq t_2$

$\phi_5 : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \longmapsto \psi(t_1 \cdot t_2^{-6})$ is not generic

Construction of supercuspidal reps à la Yu (+ twist Fintzen–Kaletha–Spice)

Input: (i) $G^0 \subsetneq G^1 \subsetneq \ldots \subsetneq G^{n-1} \subseteq G^n = G$

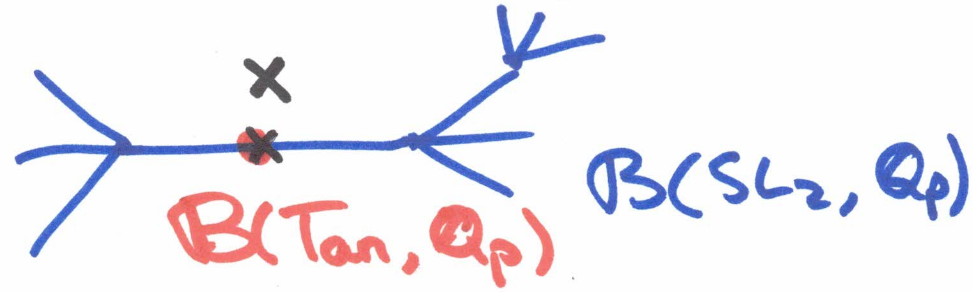
tame twisted Levi subgp s.th.

$Z(G^0)/Z(G)$ is anisotropic, i.e., compact

e.g. $G^0 = \mathrm{Tan} \subset SL_2(\mathbb{Q}_p) = G^1 = G$

(ii) $x \in \mathcal{B}(\underline{G}^0, F) \subset \mathcal{B}(\underline{G}^1, F) \subset \ldots \subset \mathcal{B}(\underline{G}, F)$

e.g.



s.th. x is a vertex in $\mathcal{B}(\underline{G}^0, F)$

(iii) $0 < r_0 < r_1 < \ldots < r_{n-1}$

(iv) ϕ_i $(0 \leq i \leq n-1)$ a (G^{i+1}, G^i)-generic of G^i of depth r_i

(v) ρ^0 an irred rep of G^0_x such that $\rho^0|_{G^0_{x,0+}}$ is trivial and $\rho^0|_{G^0_{x,0}}$ is a cuspidal rep of $G^0_{x,0}/G^0_{x,0+}$

Construction:

$\tilde{K} = G^0_x G^1_{x,r_0/2} G^2_{x,r_1/2} \cdots G^n_{x,\frac{r_{n-1}}{2}}$

$\tilde{\rho} = \rho^0 \otimes \kappa$ repr of $\tilde{K}$

$\rho^0 : \tilde{K} \to \tilde{K}/G^0_{x,0+} G^1_{x,r_0/2} \cdots$

$\simeq G^0_x/G^0_{x,0+} \xrightarrow{\rho^0} \mathrm{End}(V_{\rho^0})$

$\kappa = \kappa^{nt} \otimes \varepsilon_{FKS} \leftarrow \varepsilon_{FKS} : \tilde{K} \to G^0_x/G^0_{x,0+} \to \{\pm 1\}$

$\uparrow$ built from $\{\phi_i\}$ via theory of Heisenberg-Weil reps

(Fintzen-Kaletha-Spice)

<u>Thm</u> (Yu 2001 (Fintzen 2021) $p \neq 2$
Fintzen – Schwein 2025 $q \geq 4$)

$c\text{-}\mathrm{ind}_{\tilde{K}}^{G} \tilde{\rho}$ is irreducible supercuspidal repr

<u>Thm</u> (Kim 2007 ($p >> 0$, char $F = 0$)
(Fintzen 2021)

If $p \nmid |$Weyl group of $G|$, then <u>all</u> supercuspidal reps arise from this construction.

<u>Sketch of the construction of K</u>
in the case where $n = 1$ & $p \neq 2$

input: $(G^0 \subseteq G^1 = G, x, r_0, \phi_0, \rho^0)$
$r_0 = r$, $\phi_0 = \phi$

$\rightsquigarrow \tilde{K} = G^0_x G_{x, r/2}$

<u>Step</u> 1: Extend $\phi|_{G^0_x}$ to a character $\hat{\phi}$ of $G^0_x G_{x, r/2+}$
(by "sending root gps of G outside G^0 to 1")

Step 2 (Heisenberg repr):

"extend" $\hat{\phi}$ to a repr of $G_{x,r/2}$ as follows:

Fact: $V_{r/2} := G_{x,r/2} / G^0_{x,r/2} G_{x,r/2+}$ is $\mathbb{F}_p$-vector space and

$\langle g, h \rangle = \hat{\phi}(\underbrace{ghg^{-1}h^{-1}}_{\in G_{x,r}})$ is $g, h \in G_{x,r/2}$

a non-degenerate symplectic form on $V_{r/2}$ ($\mu_p \simeq \mathbb{F}_p$)

$\Leftarrow$ ϕ generic

Fact (Heisenberg rep): $\exists$ a unique (up to isom) irreducible rep (ω, V_ω) of $G_{x,r/2}$ s.th. $\omega|_{G_{x,r/2+}} = \hat{\phi} \cdot \mathrm{Id}$

we have $\dim V_\omega = \sqrt{\# V_{r/2}}$

<u>Step 3</u> (Weil rep): Define a compatible action of G_x° on V_ω: $G_x^\circ \curvearrowright V_{r/2}$ and preserves $\langle\ ,\ \rangle$

via conjugation

$\rightsquigarrow G_x^\circ \longrightarrow Sp(V_{r/2}) \curvearrowright V_\omega$

Weil repr $\otimes\, \phi$

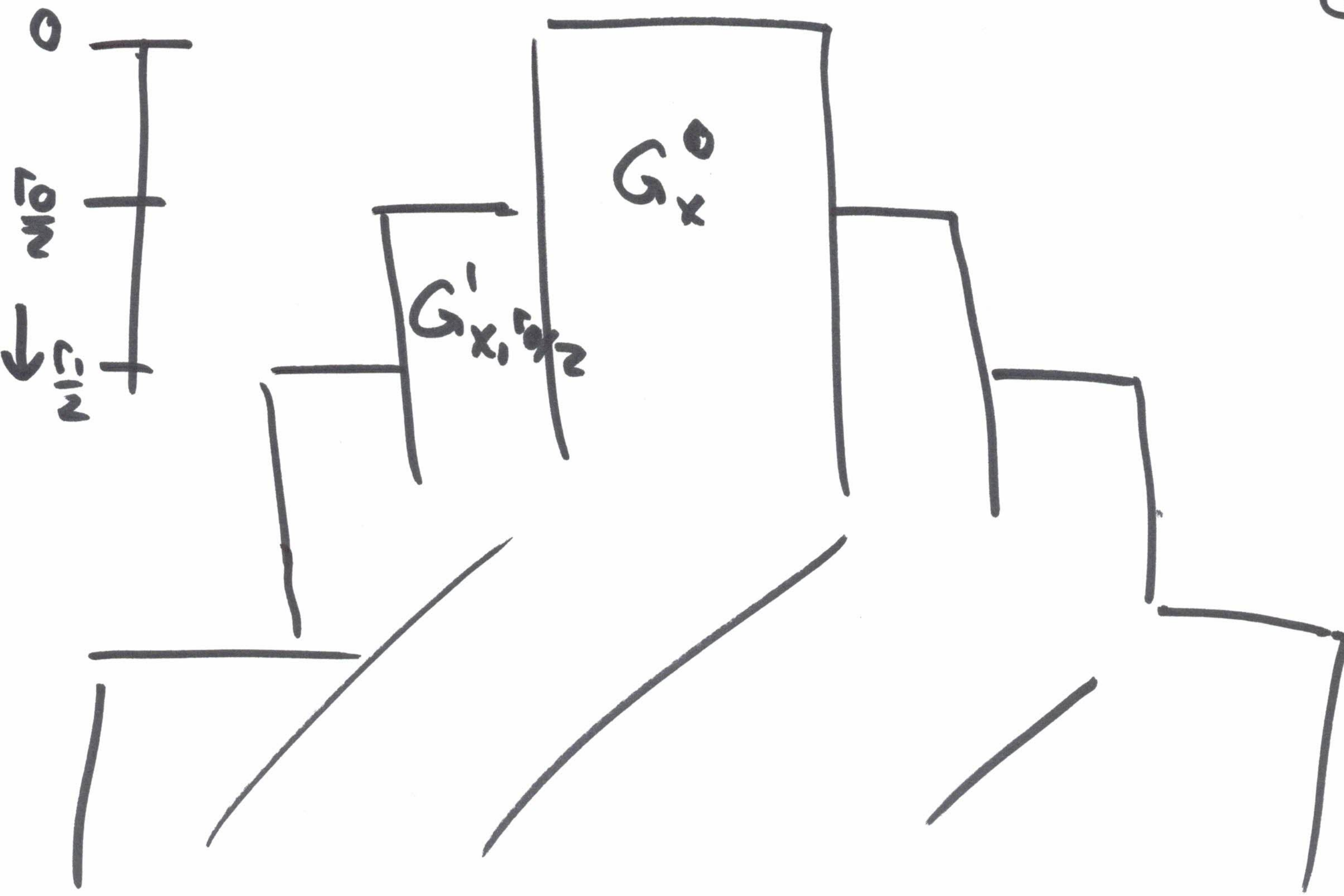
call κ^{nt} the resulting rep of $\underbrace{G_x^\circ G_{x,r/2}}_{\tilde{K}}$ on V_ω

Step 4 (twist): $\kappa = \kappa^{nt} \otimes \varepsilon_{FKS}$

Construction of κ for general inputs:

for each G^i $(1 \leq i \leq n) \rightsquigarrow \kappa_i$

$$\kappa := \bigotimes_i \kappa_i$$



$$(\pi, V) \rightsquigarrow r_{n-1} \rightsquigarrow \phi_{n-1} \rightsquigarrow G^{n-1}$$
$$\rightsquigarrow r_{n-2} \rightsquigarrow \phi_{n-2} \rightsquigarrow G^{n-2}$$
$$\rightsquigarrow \cdots \rightsquigarrow \rho^0 \text{ on } G^0_x$$