Suppose G is split,
e.g. $GL_n(F)$, $SL_n(F)$, $Sp_{2n}(F)$, ~~$SO_n(F)$~~ :

Def A <u>maximal split torus</u> of $GL_n(F)$ ($SL_n(F)$, ...) is a subgroup of the form

$$g \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} g^{-1} \text{ for some } g \in GL_n(F)$$

Moy - Prasad filtration

Def: A BT triple is a triple $(T, \{X_\alpha\}_{\alpha \in \Phi(G,T)}, x_{BT})$ consisting of

(1) $T \subset G$ a maximal split torus

(2) $X_\alpha \in \underbrace{\mathrm{Lie}(G)_\alpha}_{\alpha\text{- eigenspace}} - \{0\}$ s.th.
$\{X_\alpha\}$ form a Chevalley system

$\mathrm{Lie}(G) = \mathrm{Lie}(\underline{G})(F)$

$\mathrm{Lie}(GL_n(F)) = \mathrm{Mat}_{n \times n}(F)$

$\mathrm{Lie}(SL_n(F)) = \mathrm{Mat}_{n \times n}(F)_{\mathrm{trace}=0}$

$$\underbrace{\Phi(G,T)}_{\text{non-zero weights of } T \curvearrowright \mathrm{Lie}(G)} \subset X^*(T) := \underbrace{\mathrm{Hom}_F}_{\substack{\text{homom as groups and} \\ \text{as algebraic varieties}}}(T, F^\times)$$

e.g. $\mathrm{Hom}_F(F^\times, F^\times) = \left\{ \begin{array}{l} F^\times \longrightarrow F^\times \\ x \longmapsto x^m \end{array} \,\middle|\, m \in \mathbb{Z} \right\}$

$\simeq \mathbb{Z}$

example : $G = GL_n(F)$

$$\{X_{ij}\}_{1 \le i \ne j \le n}$$

$$\begin{array}{cc} \left(\begin{array}{ccc} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{array} \right) & - \, i \\ j & \end{array}$$

(3) $x_{BT} \in \underbrace{X_*(T)}_{\substack{\| \\ \mathrm{Hom}_F(F^\times, T) \simeq \mathbb{Z}^n \\ \| \\ (F^\times)^n}} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$

Fix a BT triple $\overbrace{(T, \{X_\alpha\}_{\alpha \in \Phi(G,T)}, x_{BT})}^{=: X}$.

Moy - Prasad filtration for T

$T_0 := \{t \in T \mid \mathrm{val}(\chi(t)) = 0 \quad \forall\, \chi \in X^*(T)\}$

$= $ maximal compact subgp of T

e.g. $G = SL_2(\mathbb{Q}_p)$, $T = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \,\middle|\, t \in \mathbb{Q}_p^\times\right\}$

$$T_0 = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \,\middle|\, t \in \mathbb{Z}_p^\times\right\}$$

$r \in \mathbb{R}_{>0}$:

$T_r := \{t \in T_0 \mid \mathrm{val}(\chi(t) - 1) \geq r \quad \forall\, \chi \in X^*(T)\}$

e.g. $G = SL_2(\mathbb{Q}_p)$, $T_r = \left\{\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} \,\middle|\, t \in 1 + p^{\lceil r \rceil}\mathbb{Z}_p\right\}$

Moy - Prasad filtration for root groups U_α

$\alpha \in \Phi(G,T) \quad F \simeq U_\alpha \leq G$

dim 1

with $\mathrm{Lie}(U_\alpha) = \mathrm{Lie}(G)_\alpha$

e.g. for GL_n:

$$U_{\alpha_{ij}} = \left\{ \begin{array}{cc} & \\ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} & i \\ j & \end{array} < GL_n(F) \right\}$$

$1 \leq i \neq j \leq n$

$$x_\alpha : F \xrightarrow{\sim} U_\alpha \quad \text{s.th.}$$

$$\begin{array}{rcl} \mathrm{Lie}(x_\alpha) : F & \longrightarrow & \mathrm{Lie}(G)_\alpha \\ \Psi & & \\ 1 & \longmapsto & X_\alpha \end{array}$$

$r \geq 0$

$$U_{\alpha,x,r} := x_\alpha\left(\varpi^{\lceil r - \alpha(x_{BT}) \rceil} \mathcal{O}\right)$$

$$x_{BT} \in X_*(T) \otimes \mathbb{R} = \mathrm{Hom}_F(F^\times, T) \otimes \mathbb{R}$$

$$\times$$

$$\alpha \in \mathrm{Hom}_F(T, F^\times)$$

$$\downarrow$$

$$\mathrm{Hom}_F(F^\times, F^\times) \otimes \mathbb{R}$$

$$\simeq \mathbb{Z} \qquad \otimes \mathbb{R}$$

$\rightsquigarrow \alpha(x_{BT}) \in \mathbb{R}$

examples : $G = SL_2(\mathbb{Q}_p)$

(a) $x_1 := \left(T = \left\{\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right\}, x_{BT}\right)$

$$x_{BT} = 0 \in \underbrace{X_*(T)}_{\simeq \mathbb{Z}} \otimes \mathbb{R} \simeq \mathbb{R}$$

$\Phi(SL_2, T) = \{\pm\alpha\}$

$\alpha : \begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} \longmapsto t^2 \quad -\alpha : \begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} \longmapsto t^{-2}$

$U_\alpha = \left\{\begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}\right\} \quad U_{-\alpha} = \left\{\begin{pmatrix} 1 & 0 \\ * & 1\end{pmatrix}\right\}$

$x_\alpha : \underset{\in F}{x} \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \in U_\alpha$

$U_{\alpha, x, r} = \begin{pmatrix} 1 & p^{\lceil r \rceil}\mathbb{Z}_p \\ 0 & 1\end{pmatrix} \quad U_{-\alpha, x, r} = \begin{pmatrix} 1 & 0 \\ p^{\lceil r \rceil}\mathbb{Z}_p & 1\end{pmatrix}$

(b) $x_2 = \left(T, \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right\}, x_{BT} = \frac{1}{4} \cdot \check{\alpha}\right)$

where $\check{\alpha} : t \longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$

then $\alpha(\check{\alpha}) : t \longmapsto t^2 \mathrel{\hat{=}} 2 \in \mathbb{Z}$

$\rightsquigarrow \alpha(x_{BT}) = \frac{1}{4} \cdot 2 = \frac{1}{2}$

$$U_{\alpha, x, r} = \begin{pmatrix} 1 & p^{\lceil r - \alpha(x_{BT}) \rceil}\mathbb{Z}_p \\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} 1 & p^{\lceil r - 1/2 \rceil}\mathbb{Z}_p \\ 0 & 1\end{pmatrix}$$

$$U_{-\alpha, x, r} = \begin{pmatrix} 1 & 0 \\ p^{\lceil r + 1/2 \rceil}\mathbb{Z}_p & 1\end{pmatrix}$$

e.g. $U_{\alpha,x,0} = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$

$$U_{-\alpha,x,0} = \begin{pmatrix} 1 & 0 \\ p\mathbb{Z}_p & 1 \end{pmatrix}$$

<u>Moy-Prasad filtration</u>: $r \in \mathbb{R}_{\geq 0}$

$G_{x,r} := \langle T_r,\ U_{\alpha,x,r} \mid \alpha \in \Phi(G,T) \rangle$

e.g. $G = SL_2(\mathbb{Q}_p)$:

(a) $G_{x_1,0} = SL_2(\mathbb{Z}_p)$

$r > 0$: $G_{x_1,r} = \begin{pmatrix} 1+p^{\lceil r \rceil}\mathbb{Z}_p & p^{\lceil r \rceil}\mathbb{Z}_p \\ p^{\lceil r \rceil}\mathbb{Z}_p & 1+p^{\lceil r \rceil}\mathbb{Z}_p \end{pmatrix}_{\det=1}$

(b) $G_{x_2,0} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}_{\det=1}$

$r > 0$ $G_{x_2,r} = \begin{pmatrix} 1+p^{\lceil r \rceil}\mathbb{Z}_p & p^{\lceil r-1/2 \rceil}\mathbb{Z}_p \\ p^{\lceil r+1/2 \rceil}\mathbb{Z}_p & 1+p^{\lceil r \rceil}\mathbb{Z}_p \end{pmatrix}_{\det=1}$

$G_{x,r+} = \bigcup_{s>r} G_{x,s} = G_{x,\text{"}r+\varepsilon\text{"}}$

$G_{x,0}$ is called <u>parahoric</u> subgroup

Some properties:

(i) $G_{x,0} \trianglerighteq G_{x,r}$

(ii) $G_{x,0}/G_{x,0+} \simeq \mathbb{F}_q$-points of a reductive group

e.g. $G = SL_2(\mathbb{Q}_p)$ (a) $G_{x_1,0}/G_{x_1,0+} \simeq SL_2(\mathbb{F}_p)$

(b) $G_{x_2,0}/G_{x_2,0+} \simeq \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{F}_p^{\times} \right\}$

(iii) $[G_{x,r}, G_{x,s}] \subseteq G_{x,r+s}$

($\Rightarrow G_{x,r}/G_{x,r+}$ abelian if $r>0$)

Bruhat-Tits building

(non-traditional def)

Def: The (reduced) Bruhat-Tits building $\mathcal{B}(\underline{G}, F)$ is as a set $\{\text{BT-triples}\}/\sim$

$x_1 \sim x_2 \iff G_{x_1,r} = G_{x_2,r} \; \forall r \in \mathbb{R}_{\geq 0}$

$\rightsquigarrow$ can write $G_{x,r}$ for $x \in \mathcal{B}(\underline{G}, F)$

Properties of $\mathcal{B}(\underline{G}, F)$:

(i) G acts on $\mathcal{B}(\underline{G}, F)$ such that for $x \in \mathcal{B}(\underline{G}, F)$:

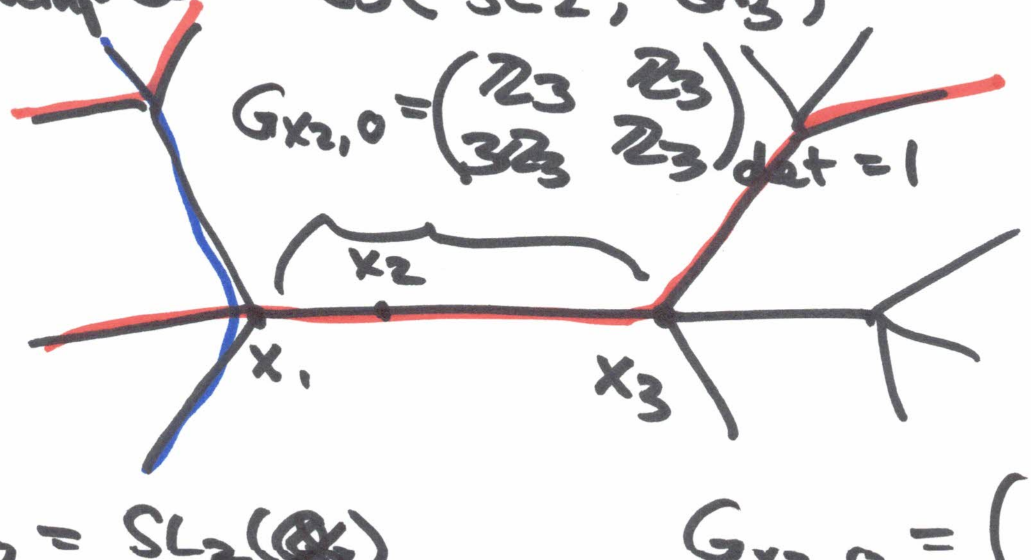
$$G_{g\cdot x, r} = g\, G_{x,r}\, g^{-1} \quad \forall r \geq 0$$

(ii) $\mathcal{B}(\underline{G}, F)$ can be equipped with a polysimplicial structure such that for $x, y \in \mathcal{B}(\underline{G}, F)$:

x and y are in the interior of the same complex

$\Leftrightarrow G_{x,0} = G_{y,0}$

example: $\mathcal{B}(SL_2, \mathbb{Q}_3)$



$G_{x_1,0} = SL_2(\cancel{\mathbb{Q}_3}\ \mathbb{Z}_3)$ $\qquad G_{x_3,0} = \begin{pmatrix} \mathbb{Z}_3 & \frac{1}{3}\mathbb{Z}_3 \\ 3\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix}$ det

apartments $\longleftrightarrow$ max split tori

$A(T, F) := \{(T, \{X_\alpha\}, x_{BT}) \mid \forall X_\alpha \ldots \forall x_{BT} \ldots$

Def Let (π, V) be a (smooth) irreducible repr of G.

The **depth** of (π, V) is the smallest non-negative real number r such that

$$V^{G_{x,r+}} \neq \{0\} \quad \text{for some } x \in \mathcal{B}(\underline{G}, F)$$

$\{v \in V \mid \pi(k)v = v \ \forall k \in G_{x,r+}\}$

The non-split:

Suppose $\underline{G} \times_F E$ is split for E/F tamely ramified

($\underline{G}$ is not nec. split)

- $r > 0$: $G_{x,r} := \underline{G}(E)_{x,r}^{Gal(E/F)}$
- $\mathcal{B}(\underline{G}, F) := \mathcal{B}(\underline{G}, E)^{Gal(E/F)}$

$A(T,F)$ is an affine space over $X_*(T)\otimes\mathbb{R} / X_*(Z(G))\otimes\mathbb{R}$

$G_{x,r}/G_{x,r+} \curvearrowright V^{G_{x,r+}} \neq \{0\}$

$G_{x,r}/G_{x,r+}$ abelian if $r>0$

not nec. ab. $r=0$