p prime, $\underbrace{F/\mathbb{Q}_p}_{<\infty}$ or $F = \mathbb{F}_q((t))$

val : $F \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$

$F \supset \mathcal{O} \ni \varpi$

$\mathcal{O}/\varpi\mathcal{O} \simeq \mathbb{F}_q$

$|\mathbb{F}_q| = q = p^r$

Def: A $\underline{\text{p-adic group}}$ is the F-points $G = \underline{G}(F)$ of a connected reductive group $\underline{G}/F$.

e.g.
- $GL_n(F)$ = $n \times n$ invertible matrices over F
- $SL_n(F) = \{ A \in GL_n(F) \mid \det A = 1 \}$
- $SO_n(F) = \{ A \in SL_n(F) \mid {}^tA A = \mathbb{1} \}$
- $Sp_{2n}(F) = \{ A \in SL_{2n}(F) \mid {}^tA J A = J \}$

$$J = \begin{pmatrix} 0 & \begin{matrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix} \\ \begin{matrix} & & -1 \\ & \cdot^{\cdot^{\cdot}} & \\ -1 & & \end{matrix} & 0 \end{pmatrix} \begin{matrix} \}n \\ \\ \end{matrix}$$

$\hspace{6em}\underbrace{\hspace{3em}}_{n}$

- products of the above

(but also exceptional)

From now on G a p-adic group

topology on F ⇝ topology on G

properties:

1) G has a basis ~~cons~~ of open neighborhoods of 1 consisting of compact open subgps:

e.g. $G = GL_n(\mathbb{Q}_p) \supset$

$$GL_n(\mathbb{Z}_p) \supset \mathbb{1} + p \cdot \mathrm{Mat}_{n \times n}(\mathbb{Z}_p)$$
$$\supset \mathbb{1} + p^2\, \mathrm{Mat}_{n \times n}(\mathbb{Z}_p)$$
$$\supset \dots$$
$$\supset \mathbb{1} + p^N\, \mathrm{Mat}_{n \times n}(\mathbb{Z}_p)$$
$$\supset \dots$$

⇝ Moy Prasad filtration

2) G is totally disconnected

$\mathbb{Z}_3$ 

C = algebraically closed field
e.g. $\mathbb{C}$, $\overline{\mathbb{F}_\ell}$, $\overline{\mathbb{F}_p}$

<u>Def</u> A <u>smooth</u> representation of G is a pair (π, V) consisting of

- a C-vector space V
- a group homomorphism $\pi : G \longrightarrow \mathrm{Aut}_C(V)$ such that $\forall\, v \in V$ $\exists$ a compact open subgp ~~of G~~ $K < G$ s.th. $\pi(k)(v) = v$ $\forall\, k \in K$

Examples:

(1) $V = C$, $\underset{\text{triv}}{\overset{\|}{\pi}} : G \longrightarrow 1 \in C^*$

(trivial representation)

(2) $G = SL_2(\mathbb{Q}_p)$, $B = \begin{pmatrix} \mathbb{Q}_p & \mathbb{Q}_p \\ 0 & \mathbb{Q}_p \end{pmatrix}_{\det = 1}$

$$\mathrm{Ind}_B^G \mathrm{triv} = V = \left\{ \begin{array}{l} f : B\backslash G \longrightarrow \mathbb{C} \mid \\ f \text{ locally constant} \end{array} \right\}$$

$$= \{ f : \mathbb{P}^1(\mathbb{Q}_p) \longrightarrow \mathbb{C} \mid f \text{ loc const} \}$$

$$\pi : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$$

$$\pi(g) : V \longrightarrow V \qquad g \in G$$

$$x \mapsto f(x) \longmapsto x \mapsto f(xg)$$

$$V = \left\{ f : G \longrightarrow \mathbb{C} \mid \begin{array}{l} \cdot\ f(bg) = f(g) \ \forall b \in B,\ g \in G \\ \cdot\ f \text{ loc constant} \end{array} \right\}$$

<u>Def</u>: A parabolic subgroup of $GL_n(F)$ ($SL_n(F)$, $SO_n(F)$, $Sp_n(F$ is a subgroup of the form

$$g \begin{pmatrix} \boxed{*}_{n_1} & & & * \\ 0 & \boxed{*}_{n_2} & & \\ & & \ddots & \\ 0 & & 0 & \boxed{*} \end{pmatrix} g^{-1}$$

for some choice of blocks ($n_1 + n_2 + \ldots + * = n$) and some $g \in GL_n(F)$ (or $SL_n(F) \ldots$)

<u>Levi decomposition:</u>

$$\underbrace{g \begin{pmatrix} \boxed{GL} & * \\ & \boxed{\cdot} & \\ 0 & & \boxed{GL} \end{pmatrix} g^{-1}}_{P} = \underbrace{g \begin{pmatrix} \boxed{*} & & 0 \\ & \boxed{*} & \\ 0 & & \boxed{*} \end{pmatrix} g^{-1}}_{\text{p-adic group called \underline{Levi subgroup}}} \ltimes \underbrace{g \begin{pmatrix} \boxed{1} & & * \\ & \boxed{1} & \\ 0 & & \boxed{1} \end{pmatrix} g^{-1}}_{\text{unipotent radical of } P}$$

$$P = M \ltimes N$$

parabolic induction: If $C = \mathbb{C}$

Let $P = M \ltimes N \subseteq G$ parabolic subgp

(σ, V_σ) a smooth rep of M

The parabolic induction is the representation $(n\text{-}\mathrm{Ind}_P^G \sigma, n\text{-}\mathrm{Ind}_P^G V_\sigma)$:

- $n\text{-}\mathrm{Ind}_P^G V_\sigma := \{ f : G \longrightarrow V_\sigma \mid$
  - $f(mng) = \delta_P^{1/2}(m)\sigma(m)(f(g))$
    $m \in M, n \in N, g \in G$
  - $\exists K_f \subset G$ compact, open subgp
    $f(gk) = f(g) \; \forall k \in K_f \}$
- $(n\text{-}\mathrm{Ind}_P^G \sigma)(g) : (x \mapsto f(x)) \longmapsto (x \mapsto f(xg))$
  $g \in G$

$(n\text{-}\mathrm{Ind}_P^G \sigma, n\text{-}\mathrm{Ind}_P^G V_\sigma)$ is a smooth representation of G.

$\delta_P(m) := |\det(\mathrm{Ad}_{\mathrm{Lie}(N)}(m))|_p$ ← p-adic absolute value

↑ modulus character

From now on assume $C = \mathbb{C}$

<u>Def</u>: A <u>supercuspidal representation</u> (π, V) of G is a (smooth) irreducible repr of G s.th.
($\Leftarrow$ has exactly two subrep! $\{0\}, V$)
$(\pi, V) \not\hookrightarrow \mathrm{Ind}_P^G V_\sigma$ for all proper parabolic subgps $P = M \ltimes N \subsetneqq G$ and all (smooth) irred reps (σ, V_σ) of M

<u>Fact</u>: (π, V) irred repr of G
Then $\exists$ a parabolic subgp $P = M \ltimes N \subseteq G$ and a supercuspidal repr (σ, V_σ) of M s.th.
$(\pi, V) \hookrightarrow (\mathrm{Ind}_P^G \sigma, \mathrm{Ind}_P^G V_\sigma)$

Bernstein decomposition:

$$\mathrm{Rep}(G) = \prod_{\{(M,\sigma)\}/\sim} \overbrace{\mathrm{Rep}(G)_{[M,\sigma]}}^{\text{Bernstein blocks}}$$

$\mathrm{Rep}(G)$: category of smooth reps of G

M: Levi subgp of a parab of G

σ: supercuspidal rep

$$(M,\sigma) \sim (gMg^{-1}, \sigma(g^{-1} - g) \otimes \chi)$$

for some $g \in G$ and some unramified character χ of gMg^{-1}

i.e. a character $\chi : gMg^{-1} \to \mathbb{C}^\times$ that is trivial on all compact subgroups of G

$\mathrm{Rep}(G)_{[M,\sigma]}$ consists of all the smooth reps whose irreducible subquotients embedd into

$$\mathrm{Ind}_{P'}^{G} \sigma', \quad P' = M' \ltimes N' \quad (M',\sigma') \sim (M,\sigma)$$

Example: $G = SL_2(F)$

(a) $M = G$, $\mathrm{Rep}(G)_{[G,\sigma]} = \{\sigma,\ \sigma\oplus\sigma,\ \sigma\oplus\sigma\oplus\sigma, \dots\}$

$$\mathrm{Hom}_G(\cancel{\sigma,\sigma}\,(\sigma, V_\sigma),(\sigma, V_\sigma)) \simeq \mathbb{C}$$

(b) $P = B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \supset M = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\} =: T$

$\mathrm{Rep}(G)_{[T,\mathrm{triv}]} =:$ principal block

$$1 \to \mathrm{triv} \longrightarrow \mathrm{Ind}_{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}^{G} \mathrm{triv} \longrightarrow \mathrm{St} \longrightarrow 1$$

called Steinberg rep

$P = M \ltimes N$

$$V \xrightarrow{\partial_M} V_N := V / \{\pi(n)v - v \mid n \in N\}$$

adjoint to Ind_P^G

$$\mathrm{Hom}_G(\sigma, \mathrm{Ind}_P^G \pi)$$

$$\simeq \mathrm{Hom}_M(\partial_M(\sigma), \pi)$$

jacquet functor