EXERCISES FOR ARIZONA WINTER SCHOOL 2025

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1. Introduction

These are exercises written for the 2025 Arizona Winter School on Representation theory of p-adic groups. The main topics covered by these exercises are the general theory of algebraic/reductive groups and Bruhat–Tits buildings.

The reader will notice that there are *far* too many exercises in here to be coverable in any five day program. This stems from the goal of including exercises that might be helpful to as broad a range of students as possible. Even so, these exercises only scratch the surface of the theory of reductive groups and Bruhat–Tits theory.

Section 2 is designed to give a fairly broad overview of the contents of the notes with background as minimal as possible. In particular, many of the general definitions are given in that section, and special cases of much of the general theory is explained there.

Section 3 contains exercises outlining almost a complete first course on algebraic groups (and in fact a large majority of the exercises are culled from [Bor69] and [Con20]). For this reason I highly suggest that it not be read straight through; it is intended mainly to be referenceable when doing exercises in later parts of the notes. Of necessity, more background in algebraic geometry is assumed here.

Section 4 is more focused on reductive groups in particular; it contains a very quick development of the root datum of a connected reductive group, a primer on Galois cohomology and forms, and a few exercises on bounded subgroups and integral models. Especially in the section on integral models, a large amount of algebraic geometry is assumed.

Section 5 briefly introduces spherical buildings before describing affine root systems and a number of applications of Bruhat–Tits buildings (which, however, are not defined there, instead being described by a number of axioms as in [KP23]). Far less algebraic geometry is assumed in this section, but a strong familiarity with the structure theory of reductive groups is assumed.

I hope that each participant will find a natural starting part among these sections, but you should also feel free to jump among them, searching for the parts that you find most interesting.

2. Examples with classical groups

In this section, we aim to illustrate a number of concepts appearing later in the special case of GL_n, which requires far less background than more general cases. We will assume a basic familiarity with the language of scheme theory, but we will try to avoid reliance on any particular theorems in algebraic geometry.

2.1. Algebraic group theory.

Definition 2.1. An *algebraic k-group G* is a finite type affine[1] k-scheme G equipped with k-morphism $m\colon G \times_k G \to G$, $i\colon G \to G$, and $e\colon \operatorname{Spec} k \to G$ such that the following diagrams

[1]Usually the term "algebraic k-group" does not assume affineness, but the only groups we consider will be affine.

commute:

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{m\times \mathrm{id}_G} & G\times G \\ \Big\downarrow{\scriptstyle \mathrm{id}_G\times m} & & \Big\downarrow{\scriptstyle m} \\ G\times G & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{i\times \mathrm{id}_G} & G\times G \\ \Big\downarrow{\scriptstyle \mathrm{id}_G\times i} & \searrow^{e} & \Big\downarrow{\scriptstyle m} \\ G\times X & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{e\times \mathrm{id}_G} & G\times G \\ \Big\downarrow{\scriptstyle \mathrm{id}_G\times e} & \searrow^{\mathrm{id}_G} & \Big\downarrow{\scriptstyle m} \\ G\times X & \xrightarrow{m} & G \end{array}$$

where $e\colon G \to G$ denotes the composition of the structure morphism $G \to \operatorname{Spec} k$ with e. Say that G is a *linear algebraic k-group* if it is moreover smooth.

Remark 2.2. Cartier's theorem [Sta, Tag 047N] shows that all algebraic k-groups are smooth if $\operatorname{char} k = 0$. This fact simplifies the general theory considerably, but it is not true in positive characteristic, so we will pay attention to smoothness issues below.

We begin with several examples. Recall that if X is a finite type k-scheme and R is a k-algebra, then $X(R)$ denotes the set of k-scheme morphisms $\operatorname{Spec} R \to X$. For example, if $X = \operatorname{Spec} A$ then $X(R) = \operatorname{Hom}_k(A, R)$. Note that if $X = G$ is an algebraic k-group, then $G(R)$ is a group in the usual sense. Conversely, we will say that such a group-valued functor F *is an algebraic group* if there is an algebraic k-group G and natural isomorphisms $G(R) = F(R)$.[2]

Exercise 2.3. (Unwinding the definitions) Show that the following are examples of linear algebraic groups:

(1) $\mathbf{G}_{\mathrm{a}} = \mathbf{A}^1$ with addition (the *additive group*),
(2) $\mathbf{G}_{\mathrm{m}} = \mathbf{A}^1 - \{0\}$ with multiplication (the *multiplicative group*),
(3) $\mathrm{GL}(V) = \mathrm{Aut}(V) - \{\det = 0\}$ with composition (i.e., matrix multiplication), where V is a finite-dimensional k-vector space,
(4) the closed subscheme $\mathrm{SL}(V)$ of $\mathrm{GL}(V)$ defined by the vanishing of $\det -1$.

If G is one of the algebraic groups above and R is a k-algebra, what is the group $G(R)$ of R-points of G? (Recall that $G(R)$ is the set of k-scheme morphisms $\operatorname{Spec} R \to G$.)

If $V = k^n$, we will write GL_n and SL_n in place of $\mathrm{GL}(k^n)$ and $\mathrm{SL}(k^n)$.

Exercise 2.4. Show that $\mathbf{G}_{\mathrm{m}}$ is connected. (But $\mathbf{G}_{\mathrm{m}}(\mathbf{R})$ is not connected!)

Exercise 2.5. Let V be a finite-dimensional k-vector space. This exercise shows that $\mathrm{SL}(V)$ is connected (for arbitrary k).

(1) If $\dim V = 2$, prove that $\mathrm{SL}(V)$ is smooth and connected by direct computation.
(2) Show that $\mathrm{SL}(V)$ acts transitively on $V - \{0\}$. If v is a nonzero vector and $V = kv \oplus V'$, then the stabilizer of v is isomorphic to $\mathrm{SL}(V') \times \operatorname{Hom}_k(V', kv)$, where $\operatorname{Hom}_k(V', kv)$ is considered as an affine space.
(3) Deduce from (1) and (2) that $\mathrm{SL}(V)$ is smooth and connected.[3]

Exercise 2.6. Let V be a finite-dimensional k-vector space, and let $B\colon V \otimes V \to k$ be a nondegenerate alternating bilinear form. Let

$$\mathrm{Sp}(V, B)(R) = \{g \in \mathrm{GL}(V)(R)\colon B(gv, gw) = B(v, w) \text{ for all } v, w \in V_R\}.$$

(1) Show that $\mathrm{Sp}(V, B)$ is an algebraic group.
(2) If $\dim V = 2$, show $\mathrm{Sp}(V, B) \cong \mathrm{SL}(V)$.
(3) Show that $\mathrm{Sp}(V, B)$ acts transitively on $V - \{0\}$.
(4) If $v \in V - \{0\}$, show that $(kv)^{\perp} = kv \oplus^{\perp} V'$ for V' such that $B|_{V'}$ is nondegenerate. Describe the stabilizer of v in terms of $\mathrm{Sp}(V', B|_{V'})$, and show by induction that it is smooth and connected.
(5) Deduce that $\mathrm{Sp}(V, B)$ is smooth and connected.

[2]This is justified by the Yoneda lemma; see Exercise 3.1.

[3]*Hint*: use induction to show that every orbit map $\mathrm{SL}(V) \to V - \{0\}$ is smooth with connected fibers.

Because pairs (V, B) are determined up to isomorphism by $2n := \dim_k V$, one often writes Sp_{2n} in place of $\mathrm{Sp}(V, B)$.

Exercise 2.7. (A bit harder) Suppose $\operatorname{char} k \neq 2$, and let V be a finite-dimensional k-vector space, and let $B \colon V \otimes_k V \to k$ be a nondegenerate symmetric bilinear form. Let

$$\mathrm{O}(V, B)(R) = \{g \in \mathrm{GL}(V)(R) \colon B(gv, gw) = B(v, w) \text{ for all } v, w \in V_R\}.$$

Show that $\mathrm{O}(V, B)$ is an algebraic group.[4] If $V = k^n$ and B is the bilinear form $B(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{i=1}^{\lceil n/2 \rceil} x_i y_{2n+1-i}$, then $\mathrm{O}(V, B)$ is usually denoted by O_n. Let $\mathrm{SO}(V, B)$ (resp. SO_n) be the connected component of the identity in $\mathrm{O}(V, B)$ (resp. O_n).

Exercise 2.8. (Uses the theory of central simple algebras) Let A be a central simple algebra over k, and let $A^\times(R) = (A \otimes_k R)^\times$.

1. Show that if $A = \mathrm{End}(V)$, then $A^\times \cong \mathrm{GL}(V)$. Deduce for general A that $A^\times$ is a connected linear algebraic group by Galois descent.[5]
2. Let $N \colon A \to k$ be the reduced norm, and use Exercise 2.5 to show that $A^{N=1}(R) = (A \otimes_k R)^{N=1}$ defines a connected linear algebraic group.

Exercise 2.9. Let A be a finite k-algebra of dimension n. For a finite type affine k-scheme X, define the *Weil restriction* $\mathrm{R}_{A/k} X$ functorially by $(\mathrm{R}_{A/k} X)(R) = X(R \otimes_k A)$.

1. If $X = \mathbf{A}^m$, show $\mathrm{R}_{A/k} X \cong \mathbf{A}^{mn}$.
2. In general, use a closed embedding $X \to \mathbf{A}^m$ to deduce that $\mathrm{R}_{A/k} X$ is a finite type k-scheme.
3. (Uses more algebraic geometry) If X is smooth, show that $\mathrm{R}_{A/k} X$ is smooth.[6]
4. Show that $\mathrm{R}_{A/k} G$ is an algebraic group.

Exercise 2.10. Let k'/k be a quadratic Galois extension, and let σ be the nontrivial element of $\mathrm{Gal}(k'/k)$. Let V be a finite-dimensional k'-vector space, and let $h \colon V \otimes_k V \to k'$ be a nondegenerate k-bilinear such that $h(cv, w) = ch(v, w)$ and $h(v, w) = \sigma(h(w, v))$ for all $c \in k'$ and $v, w \in V$. Define

$$\mathrm{U}(h)(R) = \{g \in \mathrm{GL}(V \otimes_k R) \colon h(gv, gw) = h(v, w) \text{ for all } v, w \in V_R\}.$$

Show that $\mathrm{U}(h)_{k'} \cong \mathrm{GL}(V)$ and deduce by Galois descent that $\mathrm{U}(h)$ is a connected linear algebraic group.

The algebraic group $\mathbf{G}_{\mathrm{m}}$ is an example of a *torus*. Although these form a *very* special class of algebraic groups, their good properties turn out to be the foundation of the theory of reductive groups.[7]

Definition 2.11. A *k-torus* is an algebraic k-group T such that $T_{\overline{k}} \cong \mathbf{G}_{\mathrm{m}}^r$ for some r. We say T is *split* if $T \cong \mathbf{G}_{\mathrm{m}}^r$. A *maximal* k-torus in an algebraic group G is a k-torus $T \subset G$ which is not contained in any strictly larger k-torus in G.

Exercise 2.12. Let T be the closed subscheme of GL_n (resp. SL_n) consisting of diagonal matrices.

1. Show that if R is a k-algebra and $g \in \mathrm{GL}_n(R)$ (resp. $g \in \mathrm{SL}_n(R)$) is a section such that for every R-algebra S and $t \in T(S)$, we have $gt = tg$, then $g \in T(R)$.
2. Deduce that T is a maximal torus of GL_n (resp. SL_n).

Exercise 2.13. Let S be the closed subscheme of $\mathbf{A}^2_{\mathbf{R}}$ consisting of pairs (a, b) such that $a^2 + b^2 = 1$. Show that S can be given a unique linear algebraic group structure whose multiplication is given by $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$. Prove that $S_{\mathbf{C}} \cong \mathbf{G}_{\mathrm{m}}$ (so S is an $\mathbf{R}$-torus), but $S \not\cong \mathbf{G}_{\mathrm{m}}$.[8]

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[4] In fact, $\mathrm{O}(V, B)$ is a linear algebraic group with two connected components; feel free to try to prove this.

[5] $A^\times$ is called a *form* of GL_n; see Section 4.3.

[6] *Hint*: use the infinitesimal lifting criterion.

[7] For readers versed in the theory of compact Lie groups, this should sound familiar.

[8] *Hint*: what is $S(\mathbf{R})$?

Exercise 2.14. This exercise determines the maximal subtori of GL_n over a field k. Recall that an *étale algebra* over k is a k-algebra isomorphic to $\prod_{i=1}^n k_i$ for some finite separable field extensions k_i/k.

If $T \subset \mathrm{GL}_n$ is a maximal k-torus, let A_T be the k-subalgebra of $\mathrm{Mat}_{n\times n}(k)$ generated by $T(k)$.

(1) Show that A_T is an étale k-algebra.
(2) If k is infinite, show that the map $T \mapsto A_T$ gives a bijection from the set of k-tori in GL_n to the set of étale k-subalgebras of $\mathrm{Mat}_{n\times n}(k)$ with inverse sending A to the Zariski closure of $A^\times \subset \mathrm{GL}_n(k)$.
(3) Use Galois descent to extend (2) to the case of finite k.
(4) Show that two maximal k-tori $T_1, T_2 \subset \mathrm{GL}_n$ are $\mathrm{GL}_n(k)$-conjugate if and only if A_{T_1} and A_{T_2} are isomorphic k-algebras.[9]
(5) Deduce that, if k is a separably closed field, then any two maximal k-tori in GL_n are $\mathrm{GL}_n(k)$-conjugate. However, show that $\mathrm{SL}_{2,\mathbf{R}}$ admits two maximal $\mathbf{R}$-tori which are not $\mathrm{SL}_2(\mathbf{R})$-conjugate.[10]

Recall that a morphism $f\colon X \to Y$ of k-varieties is *proper* if f is of finite type, separated (i.e., the diagonal $X \to X \times_Y X$ is a closed embedding), and universally closed (i.e., for every morphism $Z \to Y$ of k-varieties, the projection $X \times_Y Z \to Z$ is topologically closed). In particular, a k-scheme X is *proper* if $X \to \operatorname{Spec} k$ is a proper morphism.

Remark 2.15. If $k = \mathbf{C}$, then a k-morphism $f\colon X \to Y$ is proper if and only if $f(\mathbf{C})\colon X(\mathbf{C}) \to Y(\mathbf{C})$ is proper, i.e., $f(\mathbf{C})$ is a closed map and $f(\mathbf{C})^{-1}(y)$ is compact for every $y \in Y(\mathbf{C})$.

Theorem 2.16 (Valuative criterion of properness)**.** *If X is a k-scheme, then X is proper if and only if, for every smooth curve C over k, every closed point $c \in C$, and every map $f\colon C - \{c\} \to X$, there is a unique extension of f to C.*

Exercise 2.17. For integers $0 \le m \le n$, there is a scheme $\mathrm{Gr}(m, n)$ defined functorially by

$$\mathrm{Gr}(m,n)(R) = \{V \subset R^n \colon R^n/V \text{ is a finite projective } R\text{-module of rank } m\}.$$

Use the valuative criterion of properness to prove that $\mathrm{Gr}(m, n)$ is proper.

Definition 2.18. If $k = \overline{k}$ and G is a connected linear algebraic k-group, then a *Borel* k-subgroup of G is a smooth connected k-subgroup $B \subset G$ such that G/B is proper, and such that B is minimal with respect to this property. Equivalently [Con20, Theorem 22.1.1], a Borel subgroup is a maximal solvable smooth connected k-subgroup of G.

Exercise 2.19. (Requires some scheme theory) Let $G = \mathrm{GL}_n$, and let B be the closed k-subgroup of G consisting of upper-triangular matrices. Let $0 \subset V_1 \subset \cdots \subset V_n = k^n$ be the standard complete flag in k^n. Show that G/B is proper as follows:

(1) Using the Yoneda lemma (Exercise 3.1), show that there is a map $f\colon G \to \prod_{i=1}^{n-1} \mathrm{Gr}(n-i, n)$ sending g to $(gV_1, \dots, gV_{n-1})$.
(2) Show that f factors through a map $G/B \to \prod_{i=1}^{n-1} \mathrm{Gr}(n-i, n)$. (We will also denote this map by f.)
(3) Let F be the subfunctor of $\prod_{i=1}^{n-1} \mathrm{Gr}(n-i, n)$ such that for every k-algebra R, $F(R)$ consists of sequences $(W_1, \dots, W_{n-1})$ such that $W_i \subset W_{i+1}$ for all i. Show that F is representable by a closed subscheme of $\prod_{i=1}^{n-1} \mathrm{Gr}(n-i, n)$.
(4) Show that f factors through an isomorphism $G/B \to F$.[11]

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[9] *Hint*: use Skolem–Noether.
[10] *Hint*: Exercise 2.13
[11] *Hint*: Zariski's main theorem

Definition 2.20. A *parabolic* k-subgroup of G is a linear algebraic k-subgroup $P \subset G$ such that G/P is proper.

Exercise 2.21. Given a flag $\mathcal{F} = (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m = V)$ of a k-vector space V, define $P_{\mathcal{F}} \subset \mathrm{GL}(V)$ to be the algebraic k-subgroup stabilizing $\mathcal{F}$. If $\mathcal{F}$ is a complete flag, prove that $P_{\mathcal{F}}$ is a Borel k-subgroup of $\mathrm{GL}(V)$. Prove that every parabolic k-subgroup of $\mathrm{GL}(V)$ is equal to $P_{\mathcal{F}}$ for some $\mathcal{F}$. In particular, any two Borel k-subgroups of GL_n are conjugate, and every parabolic k-subgroup of GL_n is $\mathrm{GL}_n(k)$-conjugate to a group of block-upper-triangular matrices.

2.2. **Topological group theory.**

In this section, we discuss some basics on topological groups.

Definition 2.22. A *topological group* is a topological space X equipped with continuous maps $m\colon X \times X \to X$ and $i\colon X \to X$, and a point $e \in X$ such that the following diagrams commute:

$$
\begin{array}{ccc}
X \times X \times X & \xrightarrow{m \times \mathrm{id}_X} & X \times X \\
\downarrow{\scriptstyle \mathrm{id}_X \times m} & & \downarrow{\scriptstyle m} \\
X \times X & \xrightarrow{\quad m \quad} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{i \times \mathrm{id}_X} & X \times X \\
\downarrow{\scriptstyle \mathrm{id}_X \times i} & \searrow{\scriptstyle e} & \downarrow{\scriptstyle m} \\
X \times X & \xrightarrow{\quad m \quad} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{e \times \mathrm{id}_X} & X \times X \\
\downarrow{\scriptstyle \mathrm{id}_X \times e} & \searrow{\scriptstyle \mathrm{id}_X} & \downarrow{\scriptstyle m} \\
X \times X & \xrightarrow{\quad m \quad} & X
\end{array}
$$

where $e\colon X \to X$ denotes the constant map to e.

Exercise 2.23. (Topological properties are determined at e) Let X be a topological group and prove the following claims.

(1) X is locally compact if and only if e admits a compact neighborhood.
(2) X is Hausdorff if and only if $\{e\}$ is a closed subset of X.
(3) X is totally disconnected if and only if the only connected subspace of X containing e is $\{e\}$.

Definition 2.24. A topological space X is *profinite* if it is Hausdorff, compact, and totally disconnected. It is *locally profinite* if every point admits a profinite neighborhood.

Exercise 2.25. (The reason for the terminology) Let $(X_i)_{i \in I}$ be an inverse system of finite discrete topological spaces, and let $X := \varprojlim_{i \in I} X_i$, equipped with the subspace topology from $\prod_{i \in I} X_i$ (which is equipped with the product topology). Show that X is a profinite space.

Exercise 2.26. (Tricky) Let X be a locally profinite topological space.

(1) Show directly from the definitions that if U is an open neighborhood of a point $x \in X$, then there is a subset $V \subset U$ containing x which is both open and closed in X.
(2) If X is profinite, show that there exists a directed set I and a directed system $(X_i)_{i \in I}$ of finite sets indexed by I such that $X \cong \varprojlim_{i \in I} X_i$, where each X_i is given the discrete topology.

Exercise 2.27. Let F be a topological field.[12] Prove the following claims.

(1) There is a unique way of assigning a topology to $X(F)$ for every finite type F-scheme X such that (a) if $i\colon X \to Y$ is a closed embedding of finite type F-schemes then $X(F) \subset Y(F)$ is endowed with the subspace topology, and (b) if $X = \mathbf{A}^n$ then the topology on $X(F) = F^n$ is the product topology. This is called the *analytic topology*.
(2) If F is a nonarchimedean local field and X is a finite type F-scheme then $X(F)$ is locally profinite.
(3) If $f\colon X \to Y$ is a morphism of finite type F-schemes, then $f(F)\colon X(F) \to Y(F)$ is continuous.
(4) If X and Y are finite type F-schemes, then $(X \times Y)(F) = X(F) \times Y(F)$ is endowed with the product topology.

[12] We require inversion to be continuous in the unit group of a topological field.

(5) If G is an algebraic group over F, then $G(F)$ is a topological group.

For the rest of this section, let (k, ω) be a henselian (e.g., complete) discretely valued field with perfect residue field $\mathfrak{f}$ and ring of integers $\mathfrak{o}$.[13] We require ω to be a normalized valuation, i.e., it has values in $\mathbf{Z}$, and a uniformizer has valuation 1.

We say that a subgroup G of $\mathrm{GL}_n(k)$ is *bounded* if the maps $x_{ij}\colon G \to \mathbf{Z}$ sending g to $\omega(g_{ij})$ all have finite image.

Exercise 2.28. (Sanity check) Show that if $f\colon G \to H$ is a homomorphism of algebraic groups and $\Gamma \subset G(k)$ is a bounded subgroup, then $f(\Gamma)$ is a bounded subgroup of $H(k)$.

The Bruhat–Tits building of GL_n is a simplicial complex which provides a way to organize the maximal bounded subgroups of GL_n.

Exercise 2.29. (Easy) Show that if k is locally compact, then a subgroup of $\mathrm{GL}_n(k)$ is bounded if and only if it has compact closure.

Exercise 2.30. Show that if G is a bounded subgroup of $\mathrm{GL}_n(k)$, then G is conjugate to a subgroup of $\mathrm{GL}_n(\mathfrak{o})$.[14]

Remark 2.31. Note in particular that any maximal bounded subgroup of $\mathrm{GL}_n(k)$ is equal to $\mathscr{G}(\mathfrak{o})$ for some smooth $\mathfrak{o}$-model of GL_n. This is not a coincidence; see Exercise 4.56.

Exercise 2.32. (Moy–Prasad) For all $r \in \mathbf{R}_{\geq 0}$, let $\mathrm{GL}_n(k)_r$ be the subgroup of $\mathrm{GL}_n(\mathfrak{o})$ consisting of matrices M which reduce to the identity modulo $\mathfrak{p}^{\lceil r \rceil}$, where $\mathfrak{p}$ is the maximal ideal of $\mathfrak{o}$. Let $\mathrm{GL}_n(k)_{r+} = \bigcup_{s>r} \mathrm{GL}_n(k)_s$ and define $\mathfrak{gl}_n(k)_r$ and $\mathfrak{gl}_n(k)_{r+}$ similarly.

(1) Show that $\{\mathrm{GL}_n(k)_{r+}\}_{r \geq 0}$ form a neighborhood basis for 1 in $\mathrm{GL}_n(k)$.
(2) Prove the *Moy–Prasad isomorphisms*:
$$\mathrm{GL}_n(k)_0 / \mathrm{GL}_n(k)_{0+} \cong \mathrm{GL}_n(\mathfrak{f})$$
and
$$\mathrm{GL}_n(k)_r / \mathrm{GL}_n(k)_{r+} \cong \mathfrak{gl}_n(k)_r / \mathfrak{gl}_n(k)_{r+}$$
for $r > 0$.
(3) (For scheme theorists) For each $r \geq 0$, construct a smooth affine $\mathfrak{o}$-group scheme $\mathscr{G}_r$ such that $(\mathscr{G}_r)_k \cong \mathrm{GL}_n$ and $\mathscr{G}_r(\mathfrak{o}) = \mathrm{GL}_n(k)_r$.

2.3. Buildings.
In this section, we introduce the spherical and Bruhat–Tits buildings of GL_n.

Definition 2.33. A *simplicial complex* is a pair $\mathcal{S} = (V, \mathcal{F})$, where V is a set and $\mathcal{F}$ is a collection of nonempty finite subsets of V such that $\mathcal{F}$ is downward closed, i.e., if $F \in \mathcal{F}$ and $\emptyset \neq F' \subset F$, then $F' \in \mathcal{F}$. The elements of V are called *vertices*, and the elements of $\mathcal{F}$ are called *facets* of $\mathcal{S}$.

Simplicial complexes are intended as a combinatorial incarnation of "topological spaces built from triangles". To make this precise, we have the following definition.

Definition 2.34. [Gar97, §13.5] If $\mathcal{S} = (V, \mathcal{F})$ is a simplicial complex, then the *geometric realization* $|\mathcal{S}|$ is the set of non-negative real-valued functions f on V such that $\sum_{v \in V} f(v) = 1$ and such that there is some $F \in \mathcal{F}$ such that $f(v) = 0$ whenever $v \notin F$.

Exercise 2.35. Define a function $d\colon |\mathcal{S}| \times |\mathcal{S}| \to \mathbf{R}$ by $d(f, g) = \sup_{v \in V} |f(v) - g(v)|$.[15]

(1) Show that d is a metric.

[13] The point is that we want to allow finite extensions of $\mathbf{Q}_p$ and $\mathbf{F}_p(\!(t)\!)$ as well as $\mathbf{Q}_p^{\mathrm{un}}$ and $\overline{\mathbf{F}}_p(\!(t)\!)$.

[14] *Hint*: construct an $\mathfrak{o}$-lattice in k^n stabilized by G.

[15] This metric is useful for constructing a canonical topology on geometric realizations, but it is not the metric mentioned in Section 5.2.

(2) Let $V = \{0, \dots, n\}$ and let $\mathcal{F}$ be the set of all nonempty subsets of V. Show that $\Delta^n := (V, \mathcal{F})$ is a combinatorial model for an n-simplex, i.e., $|\Delta^n| \cong \{(t_0, \dots, t_n) \in \mathbf{R}_{\geq 0}^{n+1} \colon \sum_{i=0}^n t_i = 1\}$ as topological spaces.
(3) Let $\partial\Delta^n := (V, \mathcal{F} \setminus \{\{V\}\})$, and find a satisfying reason for this notation.

In particular, a facet F of $\mathcal{S}$ of cardinality $n+1$ should be thought of as an n-simplex (i.e., "an n-dimensional triangle"), and a subset $F' \subset F$ of cardinality n should be thought of as a face of F.

Definition 2.36. If $n \geq 0$ is an integer and F is a facet of a simplicial complex $\mathcal{S}$, we call $|\mathcal{F}| - 1$ the *dimension* of $\mathcal{F}$. The *dimension* of $\mathcal{S}$ is the supremum over the dimensions of its facets. If *every* maximal facet of $\mathcal{S}$ is of dimension n, we say $\mathcal{S}$ is *of pure dimension* n.

Exercise 2.37. (Easy) Show that a simplicial complex of dimension ≤ 1 is the same thing as a(n undirected) graph without loops or multiple edges.

Exercise 2.38. Let $\mathcal{S}$ be a finite simplicial complex. Show that $\mathcal{S}$ is of dimension $\leq n$ if and only if for each $x \in |\mathcal{S}|$, there is a neighborhood U of x such that U embeds as a locally closed subspace of $\mathbf{R}^n$.

Definition 2.39. Let $n \geq 2$, let k be a field, and define a simplicial complex $\mathcal{S}_n = \mathcal{S}_n(k) = (V_n, \mathcal{F}_n)$ as follows:

- V_n is the set of nontrivial proper linear subspaces of k^n,
- $\mathcal{F}_n$ is the set of flags $U_1 \subset \cdots \subset U_m$ of elements of V_n.

Exercise 2.40. (Easy) Show that $\mathcal{S}_n$ is a simplicial complex of pure dimension $n-2$.

Exercise 2.41. Let $\mathscr{L} = \{L_1, \dots, L_n\}$ be a collection of 1-dimensional linear subspaces which generate k^n. (This $\mathscr{L}$ is called a *frame*.) Let $\mathcal{A}(\mathscr{L})$ be the subcomplex of $\mathcal{S}_n$ whose vertices are the sums of elements of $\mathscr{L}$.

(1) (Easy) Draw a picture of $\mathcal{A}(\mathscr{L})$ when $n = 2, 3$.
(2) (Harder; I can't do it) Draw a picture of $\mathcal{A}(\mathscr{L})$ when $n = 4$.
(3) Prove that $|\mathcal{A}(\mathscr{L})| \cong S^{n-2}$ for all n.[16]

Exercise 2.42. (Axioms for a thin simplicial complex)

(1) Show that every $(n-3)$-dimensional facet of $\mathcal{A}(\mathscr{L})$ lies in precisely two $(n-2)$-dimensional facets.
(2) Show that any two facets of $\mathcal{S}_n$ lie in some common $\mathcal{A}(\mathscr{L})$.
(3) Show that if $\mathscr{L}$ and $\mathscr{L}'$ are two frames of k^n, then there is an automorphism of $\mathcal{S}_n$ sending $\mathcal{A}(\mathscr{L})$ isomorphically to $\mathcal{A}(\mathscr{L}')$.

Exercise 2.43. Draw $\mathcal{S}_3(\mathbf{F}_2)$.

We now move on to the Bruhat–Tits building of GL_n. For the rest of this section, let (k, ω) be a henselian discretely valued field with perfect residue field $\mathfrak{f}$ and ring of integers $\mathfrak{o}$. The Bruhat–Tits building of GL_n is a simplicial complex which provides a way to organize the maximal bounded subgroups of $\mathrm{GL}_n(k)$.

A *periodic lattice chain* is a nonempty totally ordered set L of $\mathfrak{o}$-lattices in k^n such that $\Lambda \in L$ implies $x\Lambda \in L$ for all $x \in k^\times$. Say that L is *minimal* if it does not properly contain another periodic lattice chain.

Exercise 2.44. (Easy) Show that a periodic lattice chain L is minimal if and only if there is some $\mathfrak{o}$-lattice $\Lambda \subset k^n$ such that $L = \{x\Lambda \colon x \in k^\times\}$.

[16] *Hint*: Construct a triangulation of S^{n-2} isomorphic to $\mathcal{A}(\mathscr{L})$.

Define a simplicial complex $\mathcal{B}_n = (V_n, \mathcal{F}_n)$ with vertex set V_n consisting of minimal periodic lattice chains in k^n. A set $\{L_1, \dots, L_m\}$ of minimal periodic lattice chains in k^n is a facet if $\bigcup_{i=1}^m L_i$ is a periodic lattice chain. We call $\mathcal{B}_n$ the *Bruhat–Tits building* associated to GL_n.

Exercise 2.45. (Easy) Show that $\mathcal{B}_n$ is of dimension $n-1$.

Exercise 2.46. The only building among the $\mathcal{B}_n$ that is really drawable is $|\mathcal{B}_2|$. Do it! (As you will see, it depends only on $|\mathfrak{f}|$.) Prove that it is a tree, i.e., it does not contain any loops.[17]

Exercise 2.47. Let $e_1, \dots, e_n$ be the standard k-basis of k^n, and for every $I \subset \{1, \dots, n\}$ let V_I be the span of the e_i for $i \in I$.

1. (1) Construct a natural action of $\mathrm{GL}_n(k)$ on $\mathcal{B}_n$.
2. (2) Show that for every facet $\mathcal{F}$ of $\mathcal{B}_n$, there is some sequence $I := (I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_m = \{1, \dots, n\})$ such that $\mathcal{F}$ can be translated to $\mathcal{F}_I := \{x \cdot V_{I_j} : x \in k^\times, 1 \le j \le m\}$ by an element of $\mathrm{GL}_n(k)$.
3. (3) Describe the stabilizer $\mathcal{I}_I$ of $\mathcal{F}_I$ in $\mathrm{GL}_n(k)$. Show that $\mathcal{I}_I^1 := \mathcal{F}_I \cap \{\omega(\det) = 0\}$ is bounded.
4. (4) (For scheme theorists) For a fixed I, construct a smooth affine $\mathfrak{o}$-group scheme $\mathcal{I}$ such that $\mathcal{I}_k \cong \mathrm{GL}_n$ and $\mathcal{I}(\mathfrak{o}) = \mathcal{I}_I^1$.

Remark 2.48. The Bruhat–Tits building can be understood as a geometric space which "organizes" the maximal compact subgroups of $\mathrm{GL}_n(k)$. In this sense, it is analogous to the symmetric spaces G/K associated to real or complex Lie groups G.

Exercise 2.49. If L is a periodic lattice chain, say that a basis $v_1, \dots, v_n$ of k^n is *adapted to* L if for every $\Lambda \in L$ there exist $a_1, \dots, a_n \in k^\times$ such that $\{a_1 v_1, \dots, a_n v_n\}$ is a basis for Λ. If B is a basis of k^n, let $\mathcal{A}(B)$ denote the subcomplex of $\mathcal{B}_n$ consisting of periodic lattice chains to which B is adapted.

1. (1) Show that for every periodic lattice chain L, there is a basis adapted to L.
2. (2) Show that if B is a basis of k^n, then $|\mathcal{A}(B)|$ can be considered as an affine space[18] for $\mathbf{R}^{n-1}$ in a canonical manner.
3. (3) Show that every $(n-2)$-dimensional facet of $\mathcal{A}(B)$ lies in precisely two $(n-1)$-dimensional facets.
4. (4) Show that any two facets of $\mathcal{B}_n$ lie in some common $\mathcal{A}(B)$.
5. (5) Show that if B and B' are two bases of k^n, then there is an automorphism of $\mathcal{B}_n$ sending $\mathcal{A}(B)$ isomorphically to $\mathcal{A}(B')$.

Exercise 2.50. Show that there are precisely *two* $\mathrm{SL}_2(k)$-conjugacy classes of maximal bounded subgroups of $\mathrm{SL}_2(k)$.[19]

Show that there is an element of $\mathrm{GL}_2(k)$ which swaps two adjacent vertices of $\mathcal{B}_2$, and figure out why this "explains" that there is only one maximal bounded subgroup of $\mathrm{GL}_2(k)$ (up to conjugacy).

The spherical building shows up "in" the Bruhat–Tits building in two different guises, as we will now illustrate.

Definition 2.51. If $\mathcal{S} = (V, \mathcal{F})$ is a simplicial complex and $v \in V$, then the *link* of v in $\mathcal{S}$ is the simplicial complex $\mathcal{S}_v = (V_v, \mathcal{F}_v)$, where V_v is the subset of $V - \{v\}$ consisting of vertices $w \neq v$ such that $\{v, w\} \in \mathcal{F}$, and $\mathcal{F}_v$ is the subset of $\mathcal{F}$ consisting of facets $F \in \mathcal{F}$ not containing v such that $F \cup \{v\} \in \mathcal{F}$.

Exercise 2.52. Show that the link of a vertex of Δ^n is isomorphic to Δ^{n-1}. Show that the link of a vertex of $\partial\Delta^n$ is isomorphic to $\partial\Delta^{n-1}$.

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[17] The theory of trees, centered around this example, is developed extensively in [Ser80].

[18] See Definition 5.8

[19] *Hint*: Prove that the stabilizer of a vertex v in $\mathcal{B}_2$ *only* stabilizes v, and show that two adjacent vertices do not lie in the same orbit.

Exercise 2.53. Show that the link of any vertex of $\mathcal{B}_n$ is isomorphic to $\mathcal{S}_n(\mathfrak{f})$.

Exercise 2.54. The spherical building $\mathcal{S}_n$ appears "at infinity" in $\mathcal{B}_n$. See [Gar97, §16.9] for some discussion of this point. For $\mathcal{B}_2$, this is elementary, as we show in this exercise.

Define an *end* of a tree $\Gamma = (V, E)$ to be a sequence $(v_0, v_1, \dots)$ such that v_i and v_{i+1} are joined by an edge and such that $v_i \neq v_j$ whenever $i \neq j$. (In other words, it is an infinite path with "no backtracking".) We declare that two ends $(v_0, \dots)$ and $(w_0, \dots)$ are equivalent if there is some $N \in \mathbf{Z}$ such that $v_i = w_{i+N}$ for all $i \gg 0$.

(1) Show that the $\mathrm{SL}_2(k)$-stabilizer of an end (modulo equivalence) in $\mathcal{B}_2$ is the set of k-points of a Borel k-subgroup.
(2) Construct a natural bijection between $\mathcal{S}^2(k)$ and the set of ends in $\mathcal{B}^2$.

2.4. **Decompositions.** One of the major historical motivations of Bruhat–Tits theory was to generalize the following matrix decomposition theorems. These are very useful in representation theory; see Karol Koziol's exercises for applications.

Exercise 2.55. (Bruhat decomposition) Use row reduction to show

$$\mathrm{GL}_n(F) = \coprod_{w \in P_n} B(F) w B(F),$$

for any field F, where P_n is the group of permutation matrices and $B(F)$ is the group of upper-triangular matrices in $\mathrm{GL}_n(F)$.

Exercise 2.56. (Cartan decomposition) Use the theory of modules over a principal ideal domain to show

$$\mathrm{GL}_n(k) = \coprod_{m_1 \geq m_2 \geq \cdots \geq m_n} \mathrm{GL}_n(\mathfrak{o}) \begin{pmatrix} \pi^{m_1} & & \\ & \ddots & \\ & & \pi^{m_n} \end{pmatrix} \mathrm{GL}_n(\mathfrak{o})$$

For the following exercises, let k be a discretely valued field with ring of integers $\mathfrak{o}$ and uniformizer π.

Exercise 2.57. (Iwasawa decomposition) Show that $\mathrm{GL}_n(k) = \mathrm{GL}_n(\mathfrak{o}) B(k)$, where B is as in Exercise 2.55.[20]

Exercise 2.58. (Affine Bruhat decomposition) Let $p\colon \mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(\mathfrak{f})$ be the reduction map, and let $\mathcal{I} = p^{-1}(B(\mathfrak{f}))$, where B is as in Exercise 2.55. The group $\mathcal{I}$ is called an *Iwahori subgroup*. Use Exercises 2.55 and 2.56 to show

$$\mathrm{GL}_n(k) = \coprod_{\substack{m_1, \dots, m_n \in \mathbf{Z} \\ w \in P_n}} \mathcal{I} \begin{pmatrix} \pi^{m_1} & & \\ & \ddots & \\ & & \pi^{m_n} \end{pmatrix} w \mathcal{I}$$

Exercise 2.59. (Suggested by Karol Koziol) Using the previous decompositions, prove

$$\mathrm{GL}_n(F) = \coprod_{w \in P_n} B(F) w \mathcal{I}$$

and in fact

$$\mathrm{GL}_n(F) = \coprod_{w \in P_n} B(F) w \mathcal{I}_1,$$

where $\mathcal{I}_1$ is the subgroup of $\mathcal{I}$ consisting of matrices which are *strictly* upper-triangular modulo π.

[20] *Hint*: what does this mean in term of $\mathfrak{o}$-lattices?

3. Algebraic groups

This section contains useful generalities on algebraic groups necessary to the theory of reductive groups (in fact, it essentially contains a first course on algebraic groups). We very roughly follow [Con20] and [Bor69].

We will use the language of scheme theory, although the reader with less experience in algebraic geometry (or who is only interested in characteristic 0; see Remark 2.2) should read "finite type k-scheme" as "variety over k" and "morphism of k-schemes" as "regular map of varieties over k", and so on. This being said, this section will by necessity assume more familiarity with algebraic geometry than the previous one.

3.1. **Algebraic groups.** From now on, k will be a field and G will always denote an algebraic k-group (usually linear algebraic). Many standard notions from group theory immediately adapt to this setting, provided one requires everything in sight to be algebraic. For example, a k-homomorphism $f\colon G \to H$ of algebraic groups is a morphism of k-schemes which respects multiplication, inverses, and identity sections. There are similar extensions for subgroups, actions, etc.

Exercise 3.1. (Yoneda lemma; deep but easy to prove) Let $\mathcal{C}$ be a category. If $X \in \mathcal{C}$ is an object, let h_X be the contravariant functor defined by $h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.[21] If F is another contravariant functor and $\mathrm{Hom}(h_X, F)$ is the set of natural transformations from h_X to F, prove that the map $\mathrm{Hom}(h_X, F) \to F(X)$ defined by $f \mapsto f(\mathrm{id}_X)$ is a bijection.

Exercise 3.2. Use the Yoneda lemma to show that G is equivalent to the data of

- a group $G(R)$ for every k-algebra R,
- a group homomorphism $G(R) \to G(S)$ for every homomorphism $R \to S$ of k-algebras.

Deduce that if H is another algebraic k-group and $f\colon G \to H$ is a morphism of k-schemes, then f is a homomorphism if and only if f respects multiplication. (In other words, inverses and the identity section come along for free.) Can you see this without the Yoneda lemma?[22]

Exercise 3.3. (Sanity checks for scheme theorists)

(1) Prove that G is separated.[23]
(2) If k is perfect, prove that G is smooth if and only if it is reduced.[24] Deduce that G_{red} is a smooth closed k-subgroup of G in this case.
(3) Prove that G is connected if and only if it is geometrically irreducible.[25]
(4) Let G^0 be the connected component of G containing the identity. Show that G^0 is an open and closed k-subgroup of G.
(5) If $f\colon G \to H$ is a homomorphism of algebraic groups, prove that $f(G)$ is closed in H.[26] Deduce that if f is monic then it is a closed embedding.[27]

We will rapidly review the basic theory of algebraic groups found in, for instance, [Bor69], [Con20], and [Con22].

Exercise 3.4. If $f\colon G \to H$ is a k-homomorphism, show that $(\ker f)(R) = \ker f(R)$ is an algebraic group. Prove something similar for stabilizers of points under group actions.[28]

[21] We say that h_X is *represented by* X.
[22] Personally, I have never tried.
[23] *Hint*: realize the diagonal as a base change of the identity section.
[24] *Hint*: use generic smoothness and homogeneity over $\overline{k}$.
[25] *Hint*: show that connected varieties with rational points are geometrically connected, and use (1).
[26] *Hint*: use Chevalley's theorem on constructible sets and note that if K is a k-subgroup of H, then so is its closure.
[27] *Hint*: Zariski's main theorem
[28] *Hint*: realize these as certain fiber products.

Exercise 3.5. Show that a representation $G \to \mathrm{GL}(V)$ is equivalent to a k-linear morphism $\alpha^\#\colon V \to V \otimes_k k[G]$ such that $(\alpha^\# \otimes \mathrm{id}_{k[G]}) \circ \alpha^\# = (\mathrm{id}_V \otimes m_G^\#) \circ \alpha^\#$. (Here $m_G^\#\colon k[G] \to k[G] \otimes_k k[G]$ is the map dual to the multiplication map $m_G\colon G \times G \to G$.)

Exercise 3.6. (Surpisingly useful) Algebraic groups admit faithful representations. More strongly, if $H \subset G$ are algebraic groups, prove that there is a faithful representation $G \to \mathrm{GL}(V)$ and a line $L \subset V$ such that $H = \mathrm{Stab}_G(L)$.[29]

Exercise 3.7. Using the ideas in Exercise 3.6, show that every finite type k-scheme X with a G-action embeds equivariantly into some affine space with a linear action of G.

Exercise 3.8. Let G be a connected 1-dimensional linear algebraic group over $k = \overline{k}$. We will prove G is isomorphic to one of $\mathbf{G}_\mathrm{m}$ or $\mathbf{G}_\mathrm{a}$ as follows.

1. Let X be the unique smooth connected projective curve containing G as an open subscheme. Show that the translation action of G on itself extends to an action of G on X, and deduce that $X \cong \mathbf{P}^1$.[30]
2. Using the structure of the automorphism group of $\mathbf{P}^1$, prove that $\mathbf{P}^1 - G$ consists of either 1 or 2 points.
3. Show that $\mathbf{G}_\mathrm{m}$ admits a unique group structure with identity 1, and $\mathbf{G}_\mathrm{a}$ admits a unique group structure with identity 0.

Exercise 3.9. Let $k = \mathbf{F}_p(t)$ for an odd prime p. Define a closed subgroup $G \subset \mathbf{G}_\mathrm{a}^2$ by $x^p = y - ty^p$. Show that $G_{\overline{k}} \cong (\mathbf{G}_\mathrm{a})_{\overline{k}}$ but $G(k) = \{0\}$.

Exercise 3.10. (Tricky and for scheme theorists) If G is an algebraic k-group and $H \subset G$ is a locally closed subscheme, define the *centralizer* $Z_G(H)$ by

$$Z_G(H)(R) = \{g \in G(R)\colon gh = hg \text{ for all } R\text{-algebras } S \text{ and all } h \in H(S)\}.$$

If H is smooth, show that $Z_G(H)$ is representable by a closed subscheme of G as follows:

1. Show that $Z_G(H)$ is representable by a closed subscheme of G if H is a single point in G.
2. Use Galois descent to reduce to the case that k is separably closed.
3. Show that if k is a separably closed field and X is a smooth k-scheme, then $X(k)$ is Zariski-dense in X.[31]
4. Conclude by showing that $Z_G(H) = \bigcap_{h \in H(k)} Z_G(\{h\})$.

As a special case, the *center* $Z(G)$ of G is the centralizer $Z_G(G)$.

Exercise 3.11. If G is an algebraic k-group and $H \subset G$ is a locally closed subscheme, define the *normalizer* $N_G(H)$ by

$$N_G(H)(R) = \{g \in G(R)\colon gH_Rg^{-1} = H_R\}.$$

Following the strategy of Exercise 3.10, show that $N_G(H)$ is representable by a closed subscheme of G.

3.2. **Tori.** In this section, we study properties of tori; see Definition 2.11.

Exercise 3.12. Show that representations $\mathbf{G}_\mathrm{m}^r \to \mathrm{GL}(V)$ are equivalent to $\mathbf{Z}^r$-gradings on V. In particular, representations of $\mathbf{G}_\mathrm{m}^r$ are semisimple.[32]

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[29] *Hint*: using Exercise 3.5, find a finite-dimensional G-representation $V_0 \subset k[G]$ containing a collection of generators for the ideal of $k[H]$. Let $W = V_0 \cap I$, and show $H = \mathrm{Stab}_G(W)$. Show that $L \coloneqq \bigwedge^{\dim W} W \subset V \coloneqq \bigwedge^{\dim W} V_0$ works.

[30] *Hint*: use Hurwitz's theorem and the fact that elliptic curves have only finitely many automorphisms.

[31] *Hint*: reduce to showing that $X(k)$ is nonempty, and then use a definition of smoothness to reduce to the case that X is open in an affine space.

[32] *Hint*: such a representation induces a map $V \to V \otimes k[\mathbf{G}_\mathrm{m}^r]$; what can you say about it?

Exercise 3.13. Show that if R is a k-algebra with no nontrivial idempotent elements and $\alpha\colon (\mathbf{G}_{\mathrm{m}})_R \to (\mathbf{G}_{\mathrm{m}})_R$ is an R-endomorphism, then it is equal to the n-power map $[n]$ for some n.[33] Deduce that if G is a connected linear algebraic k-group and $T \subset G$ is a normal k-subgroup which is a k-torus, then T is central in G.

Exercise 3.14. Prove that if k is a separably closed field and T is a k-torus, then $T \cong \mathbf{G}_{\mathrm{m}}^r$ for some r.[34]

Exercise 3.15. Let T be a k-torus and let $S \subset T$ be a smooth connected closed k-subgroup. Show that S is also a k-torus.[35]

Definition 3.16. If T is a k-torus, define $X^*(T)$ (resp. $X_*(T)$), the *character lattice* (resp. *cocharacter lattice*) to be the set of k-homomorphisms $T \to \mathbf{G}_{\mathrm{m}}$ (resp. $\mathbf{G}_{\mathrm{m}} \to T$), equipped with an abelian group structure via $(\lambda + \mu)(t) = \lambda(t)\mu(t)$.[36]

Exercise 3.17. Let T be a k-torus. Show that $X^*(T)$ and $X_*(T)$ are free $\mathbf{Z}$-modules of the same rank r (called the *split rank* of T), and $r = \dim T$ if and only if T is split.[37]

Exercise 3.18. (Tori are anti-equivalent to Galois lattices) Let k be a field, and let k_s be a separable closure of k. If T is a k-torus, construct a natural continuous action of $\mathrm{Gal}(k_s/k)$ on $X^*(T_{k_s})$. Prove:

(1) if Λ is a finite free $\mathbf{Z}$-module equipped with a continuous action of $\mathrm{Gal}(k_s/k)$, then there is a k-torus $\mathrm{D}_k(\Lambda)$ representing the functor $R \mapsto \mathrm{Hom}(\Lambda, (R \otimes_k k_s)^\times)^{\mathrm{Gal}(k_s/k)}$.[38] Prove $\Lambda \cong X^*(\mathrm{D}_k(\Lambda)_{k_s})$ as abelian groups with Galois action.
(2) Prove $T \cong \mathrm{D}_k(X^*(T_{k_s}))$ naturally, and thus the category of k-tori is anti-equivalent to the category of Galois lattices.
(3) a morphism $f\colon S \to T$ is surjective if and only if $f^*\colon X^*(T_{k_s}) \to X^*(S_{k_s})$ is injective.
(4) f is monic if and only if f^* is surjective.
(5) if k'/k is a finite separable field extension and T' is a k'-torus, then $X^*((\mathrm{R}_{k'/k}T')_{k_s}) \cong \mathrm{Ind}_{\mathrm{Gal}(k_s/k')}^{\mathrm{Gal}(k_s/k)} X^*(T'_{k_s})$ as Galois lattices.

The following theorem should be familiar when $G = \mathrm{GL}(V)$; however, deducing it for more general V is trickier.

Theorem 3.19 (Jordan decomposition)**.** *Suppose k is perfect. If $x \in G(k)$, then there is a unique decomposition $x = x_s x_u$ for commuting elements $x_s, x_u \in G(k)$ such that for any representation $i\colon G \to \mathrm{GL}_n$, the element $i(x_s)$ is semisimple (i.e., is diagonalizable over $\overline{k}$) and $i(x_u)$ is unipotent (i.e., has characteristic polynomial $(X-1)^n$).*

Exercise 3.20. Show that if $f\colon G \to H$ is a k-homomorphism and $x \in G(k)$, then $f(x_s) = f(x)_s$ and $f(x_u) = f(x)_u$.

Definition 3.21. We say that an element $x \in G(k)$ is *semisimple* (resp. *unipotent*) if $x = x_s$ (resp. $x = x_u$) in Theorem 3.19.

Exercise 3.22. Show that if G is connected and commutative and $G(\overline{k})$ consists of semisimple elements, then G is a torus.[39]

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[33] *Hint*: look at the map on coordinate rings.

[34] *Hint*: use Exercises 3.8 and 3.13.

[35] *Hint*: reduce to $k = \overline{k}$ and study the ideal defining S.

[36] If this is the first time you are seeing this, it may seem unnatural to mix additive and multiplicative notation in this way. However, I assure the reader that this is completely standard.

[37] *Hint*: show that the natural pairing $X^*(T) \otimes_{\mathbf{Z}} X_*(T) \to \mathbf{Z}$ is perfect.

[38] *Hint*: use Galois descent.

[39] *Hint*: choose a faithful representation and use Exercise 3.15.

3.3. **Solvable groups.** The notion of short exact sequence extends to algebraic groups in an initially surprising manner. Say that a sequence $1 \to N \to G \to Q \to 1$ of algebraic k-groups is a *short exact sequence* if $G \to Q$ is faithfully flat and N is the kernel (see Exercise 3.4).

Exercise 3.23. This exercise partially explains the point of "faithfully flat" in the definition of short exact sequence.

(1) If Q is smooth and $G \to Q$ is surjective, then it is faithfully flat.[40]
(2) (For scheme theorists) Show that $G \to Q$ is faithfully flat if and only if it is a surjection of fppf sheaves.

Exercise 3.24. Show that the n-power map $[n]\colon \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ is faithfully flat for all $n \neq 0$.[41] Its kernel is called μ_n, the scheme of nth roots of unity. Show that every proper subgroup of $\mathbf{G}_{\mathrm{m}}$ is isomorphic to μ_n for some n.

Exercise 3.25. Suppose there is a short exact sequence $1 \to T \to G \to Q \to 1$, where T and Q are k-tori.

(1) Show that T is central in G.[42]
(2) Show that the commutator map $G \times G \to G$, $(g, h) \mapsto ghg^{-1}h^{-1}$ factors through a bimultiplicative map $Q \times Q \to T$, and deduce that it is constant.
(3) Use Exercise 3.22 to deduce that G is a torus.

Exercise 3.26. Suppose there is a short exact sequence $1 \to \mathbf{G}_{\mathrm{a}} \to G \to \mathbf{G}_{\mathrm{m}}^r \to 1$.

(1) Show that there is no nontrivial algebraic group which is a closed subgroup both of $\mathbf{G}_{\mathrm{a}}$ and of a k-torus.[43]
(2) If k is not an algebraic extension of a finite field, show that there is some $t \in \mathbf{G}_{\mathrm{m}}^r(k)$ such that $\langle t \rangle$ is dense in $\mathbf{G}_{\mathrm{m}}^r$.
(3) Show that the map $G(k) \to \mathbf{G}_{\mathrm{m}}^r(k)$ is surjective.[44]
(4) In the setting of (2), let $\widetilde{t} \in G(k)$ lift t, and let T be the Zariski closure of $\langle \widetilde{t} \rangle$ in G. Show that $T \to \mathbf{G}_{\mathrm{m}}^r$ is an isomorphism and deduce $G = T \ltimes \mathbf{G}_{\mathrm{a}}$.
(5) (Requires considerably more scheme theory) For general k, show that a maximal $k(X)$-torus of $G_{k(X)}$ spreads out to a fiberwise maximal torus of G_A for some finite type integral domain A over k with fraction field $k(X)$, and deduce from (4) that $G \cong \mathbf{G}_{\mathrm{m}}^r \ltimes \mathbf{G}_{\mathrm{a}}$ if k is infinite (even if it is an algebraic extension of a finite field).

The following theorem shows that there are many short exact sequences.

Theorem 3.27 (Existence of quotients)**.** *Let $H \subset G$ be a closed k-subgroup. There is a right H-invariant map $\pi\colon G \to G/H$ such that for every right H-invariant map $f\colon G \to X$, there is a unique map $\overline{f}\colon G/H \to X$ such that $f = \overline{f} \circ \pi$. Moreover, π is faithfully flat and the map $G \times H \to G \times_{G/H} G$, $(g, h) \mapsto (gh, g)$ is an isomorphism. If H is normal in G, then G/H has a unique group structure such that π is a k-homomorphism.*

Exercise 3.28. Define $\mathrm{PGL}_n = \mathrm{GL}_n / Z$, where Z is the group of scalar matrices in GL_n.

(1) Show that $\mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/k^\times$ for any field k.[45]
(2) (Uses étale cohomology) If R is a Dedekind domain with nontrivial 2-torsion in its class group, show that $\mathrm{GL}_2(R) \to \mathrm{PGL}_2(R)$ is not surjective.[46]

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[40] *Hint*: use generic flatness and a translation argument over $\overline{k}$.
[41] *Hint*: what is the map on coordinate rings?
[42] *Hint*: use Exercise 3.13.
[43] *Hint*: use Theorem 3.19 to reduce to proving that μ_p is not a subgroup of $\mathbf{G}_{\mathrm{a}}$.
[44] *Hint*: use Hilbert 90.
[45] *Hint*: Hilbert 90.
[46] *Hint*: if I and J are ideals of R, show that $I \oplus J \cong R \oplus IJ$ as R-modules, and deduce that the map $\mathrm{H}^1(\operatorname{Spec} R, \mathbf{G}_{\mathrm{m}}) \to \mathrm{H}^1(\operatorname{Spec} R, \mathrm{GL}_2)$ kills 2-torsion.

(3) Show that $\mathrm{PGL}_n \cong \mathrm{SL}_n / \mu_n$, where $\mu_n = Z \cap \mathrm{SL}_n$, although $\mathrm{SL}_n(k) \to \mathrm{PGL}_n(k)$ is surjective if and only if every element of k is an nth power.

Definition 3.29. A connected linear algebraic group G is *k-split solvable* if there is a filtration $0 = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that G_{i-1} is normal in G_i and G_i/G_{i-1} is either isomorphic to $\mathbf{G}_\mathrm{a}$ or $\mathbf{G}_\mathrm{m}$ for all $1 \leq i \leq n$. G is *solvable* if $G_{\overline{k}}$ is $\overline{k}$-split solvable.

Exercise 3.30. Let B be the group of upper-triangular matrices in GL_n, and show that B is k-split solvable.

Motivated somewhat by Exercise 3.22, we may make the following definition.

Definition 3.31. A linear algebraic k-group U is *unipotent* if $U(\overline{k})$ consists of unipotent elements.

Exercise 3.32. Let G be a k-split solvable group, where k is an algebraically closed field, and let U be the Zariski closure of the set of unipotent elements of $G(k)$. Show that U is a normal connected unipotent k-subgroup of G. Deduce from Exercise 3.26 that $G \cong T \ltimes U$ for a maximal k-torus $T \subset G$.

Exercise 3.33. It is a general fact that unipotent groups are solvable [Con20, Example 17.2.6]. Prove this for a unipotent group U over a field k of characteristic 0 as follows:

(1) Let $u \in U(k)$ be a non-identity element. Prove that $\overline{\langle u \rangle} \cong \mathbf{G}_\mathrm{a}$, where $\overline{\langle u \rangle}$ is the Zariski closure of the subgroup of $U(k)$ generated by u.[47] Deduce that U is connected.
(2) Induct on $\dim U$.

Exercise 3.34. Let U be a 1-dimensional unipotent group over a field k, and suppose that there is a nontrivial action of $\mathbf{G}_\mathrm{m}$ on U through group automorphisms. Show that $U \cong \mathbf{G}_\mathrm{a}$ as follows:[48]

(1) Suppose there is a non-identity element $u \in U(k)$. Show that the map $\varphi\colon \mathbf{G}_\mathrm{m} \to U$, $t \mapsto t \cdot u$ surjects onto $U - \{1\}$.[49] Reduce thereby to the case that φ is monic.
(2) In the setting of (1), show that either φ or $t \mapsto \varphi(t^{-1})$ extends to a k-group isomorphism $\mathbf{G}_\mathrm{a} \to U$.

Exercise 3.35. Show that a commutative connected linear algebraic group is solvable.[50]

Exercise 3.36. Show that if G is a 2-dimensional connected linear algebraic group over a field k, then G is solvable.[51]

We'll establish a bit more about the structure of k-split solvable groups in the next section.

Theorem 3.37 (Grothendieck). *There exists a k-torus $T \subset G$ such that T_K is a maximal K-subtorus of G_K for every extension field K/k. Such a T is called a maximal torus.*

Theorem 3.37 is difficult, especially for imperfect k.

Exercise 3.38. Prove Theorem 3.37 for finite $k \cong \mathbf{F}_q$ as follows:

(1) Show that there is a *Frobenius morphism* $F\colon G \to G$ which is the identity on points and induces the map $f \mapsto f^q$ on coordinate rings. What does $F\colon \mathrm{GL}_n(\overline{k}) \to \mathrm{GL}_n(\overline{k})$ look like in coordinates?
(2) Show that F induces the trivial map on tangent spaces.

---

[47] *Hint*: choose an embedding of G into GL_n, and note that $u - 1$ is a nilpotent matrix. There is an exponential map for nilpotent matrices!

[48] Of course, by Exercise 3.8, this is only interesting if k is imperfect.

[49] *Hint*: reduce to $k = \overline{k}$ and use Exercise 3.8.

[50] *Hint*: reduce to $k = \overline{k}$, and show that the group is the product of a torus and a unipotent group.

[51] *Hint*: reduce to $k = \overline{k}$ and use Exercises 3.35 and 3.8.

(3) We can assume $G = G^0$. Using Galois descent, reduce to showing that for each $x \in G(\overline{k})$ there is some $g \in G(\overline{k})$ such that $gF(g^{-1}) = x$.
(4) Define an action of G on itself by $g \cdot x = gxF(g^{-1})$. Show that each orbit map $G_{\overline{k}} \to G_{\overline{k}}$ is smooth, hence open, and use this to conclude.

Exercise 3.39. Suppose k is not an algebraic extension of a finite field. Use Exercises 3.25 and 3.26 to show that if G is k-split solvable then Theorem 3.37 holds for G.

Theorem 3.40 (Conjugacy of maximal tori)**.** *If $k = \overline{k}$, then any two maximal k-tori of G are $G(k)$-conjugate.*

Exercise 3.41. Show that Theorem 3.40 holds if G is k-split solvable.[52]

Exercise 3.42. Theorem 3.40 can be improved considerably to the statement that, over any field k, any two maximal *split* k-tori of G are $G(k)$-conjugate [Con22, Theorem 7.1.1]. Using Exercise 2.12, prove this when $G = \mathrm{GL}_n$, and deduce it for PGL_n.

Definition 3.43. The *rank* of a linear algebraic k-group G is the dimension of a maximal torus of G. By Theorems 3.37 and 3.40, the rank is a well-defined non-negative integer.

Exercise 3.44. Show that a connected linear algebraic k-group has rank 0 if and only if it is unipotent.[53]

Definition 3.45. Say G is *split* if there is a split maximal k-torus $T \subset G$.

It can happen that G_E is split for some field extension E/k, although there is no maximal k-torus $T \subset G$ such that T_E is split.

Exercise 3.46. (Serre) Let k be a local field, let D be a division quaternion algebra over k, and let $G = D^{N=1}$ as in Exercise 2.8. Show:

(1) There exists a Galois extension E/k of even degree such that $\mathrm{Gal}(E/k)$ does not contain any subgroup of index 2.
(2) If E/k is as in (1), then $\mathrm{SL}(D)_E$ is split, but there is no maximal k-torus $T \subset \mathrm{SL}(D)$ such that T_E is split.

3.4. **Borel subgroups.** This section contains the main basic theorems on Borel subgroups (Definition 2.18).

Theorem 3.47 (Borel fixed point theorem)**.** *Let H be a k-split solvable group acting on a proper k-scheme X, and assume that $X(k) \neq \emptyset$. There is a point $x \in X(k)$ such that x is fixed by H.*

Exercise 3.48. This exercise proves the Borel fixed point theorem.

(1) Prove Theorem 3.47 if $H = \mathbf{G}_{\mathrm{a}}$ or $\mathbf{G}_{\mathrm{m}}$.[54]
(2) Using (1) and the definition of k-split solvability, deduce Theorem 3.47 in general.

Exercise 3.49. (For use in Exercise 4.11) This argument refines Theorem 3.47 to show that if $H = \mathbf{G}_{\mathrm{m}}$ and X embeds G-equivariantly into $\mathbf{P}(V)$ for some finite-dimensional G-representation V[55], then there are $\geq 1 + \dim X$ fixed points for the action of H on X.

(1) Prove the result when $\dim X = 1$.[56]

---

[52] *Hint*: reduce to the case $G = T \ltimes \mathbf{G}_{\mathrm{a}}$ for some action of T on $\mathbf{G}_{\mathrm{a}}$.
[53] *Hint*: show that every Borel subgroup contains a maximal torus.
[54] *Hint*: use the valuative criterion of properness and the embedding $H \subset \mathbf{P}^1$.
[55] By [Bri18, Theorem 5.2.1], this is automatic if X is normal and projective.
[56] *Hint*: use the same argument from Exercise 3.48.

(2) Reduce to the case that X is not contained in any hyperplane of $\mathbf{P}(V)$. Show that there is a G-stable hyperplane $H \subset \mathbf{P}(V)$ such that $X - H$ contains a nontrivial fixed point, and use induction to conclude.[57]

Exercise 3.50. (Easy) If $k = \overline{k}$, prove that Borel k-subgroups of G exist.

Exercise 3.51. If k is perfect and U is unipotent and connected, show that U is k-split solvable.

Exercise 3.52. Use the Borel fixed point theorem and Exercise 2.19 to show that if G is k-split solvable then every embedding $i\colon G \to \mathrm{GL}_n$ factors through a $\mathrm{GL}_n(k)$-conjugate of B. Deduce that $G = T \ltimes U$ for a maximal torus $T \subset G$ and a unipotent k-subgroup $U \subset G$. Moreover, representations of unipotent linear algebraic groups have nonzero fixed points.

Exercise 3.53. Show that if $T \subset G$ is a torus, then $N_G(T)/Z_G(T)$ is a finite étale k-group scheme, which is constant if T is split.[58] The group $W = N_G(T)$ is called the *Weyl group* of (G, T). If G is k-split solvable and $T \subset G$ is a maximal k-torus, show that $W = 1$.[59]

Theorem 3.54. *If $k = \overline{k}$, then a smooth connected subgroup $B \subset G$ is a Borel subgroup if and only if B is a maximal k-split solvable subgroup of G. Furthermore, any two Borel k-subgroups of G are $G(k)$-conjugate.*

Exercise 3.55. Prove Theorem 3.54 when $G = \mathrm{GL}_n$.[60]

Exercise 3.56. (Hard) Prove Theorem 3.54 for general G as follows:

(1) Choose an embedding $i\colon G \to \mathrm{GL}_n$, let B_0 be a maximal k-split solvable subgroup of G, and let B be a maximal k-split solvable subgroup of GL_n containing $i(B_0)$. Show that the natural map $G/B_0 \to \mathrm{GL}_n/B$ is a closed embedding, and deduce that B_0 is a Borel k-subgroup of G.
(2) Let B be a Borel k-subgroup of G, and suppose that B_0 is a maximal k-split solvable subgroup of G. Using Theorem 3.47, show that some $G(k)$-conjugate of B_0 is contained in B. Using (1), deduce that B_0 is $G(k)$-conjugate to B.

Exercise 3.57. Use Theorem 3.54 and Exercise 3.41 to prove Theorem 3.40 for general G.

Exercise 3.58. (Richardson) Let X be a finite type scheme over $k = \overline{k}$ equipped with an action of G. If $H \subset G$ is a closed subgroup, $Y \subset X$ is an H-stable subscheme, $y \in Y(k)$ has smooth G-stabilizer, and $T_Y(z) \cap \mathfrak{g}(z) = \mathfrak{h}(z)$ for all $z \in (G \cdot y \cap Y)(k)$ (where, e.g., $\mathfrak{g}(z)$ refers to the image of $\mathfrak{g}$ under the map $\mathfrak{g} \to T_X(z)$ induced by the orbit map), then $G \cdot y \cap Y$ is a finite union of open and closed H-orbits.[61]

Exercise 3.59. Let $S \subset H \subset G$ be k-subgroups such that S is linearly reductive (Definition 4.1 and H is smooth. Using Exercise 3.58, show that the map $Z_G(S)/Z_H(S) \to (G/H)^S$ is an isomorphism onto a connected component of $(G/H)^S$. In particular, if H is a Borel (resp. parabolic) subgroup of G, then $Z_H(S)$ is a Borel (resp. parabolic) subgroup of $Z_G(S)$.

Exercise 3.60. (Chevalley) If $B \subset G$ is a Borel, show that $N_{G(k)}(B) = B(k)$ as follows:

(1) Reduce to the case that G has trivial center and $k = \overline{k}$.
(2) Let $T \subset B$ be a maximal torus, and use Theorem 3.40 to reduce to showing $N_{G(k)}(B, T) = T(k)$.

---

[57] *Hint*: use Exercise 3.12, and take H to be defined by the vanishing of a projective coordinate with minimal weight.

[58] *Hint*: use Exercise 3.13.

[59] *Hint*: use Exercise 3.52.

[60] *Hint*: use Theorem 3.47 and Exercises 2.19.

[61] *Hint*: Show that each orbit map $H \to G \cdot y \cap Y$ is smooth, hence open.

(3) Let $n \in N_{G(k)}(B,T)$ and let $S = Z_T(n)$. If $S \neq 1$, use Exercise 3.59 and induction on $\dim G$ to show $n \in T(k)$.
(4) If $S = 1$, show $G = B$.[62]

Remark 3.61. The argument of Exercise 3.60 works almost verbatim to show that $N_{G(R)}(B_R) = B(R)$ for any finite k-algebra R, and this is enough to show $N_G(B) = B$ scheme-theoretically. For the necessary extra input, see [Con14, Corollary B.2.6].

Exercise 3.62. Let $T \subset B \subset G$ be a torus inside a Borel. Using Exercise 3.60 and Exercise 3.53, show that the map $w \mapsto wBw^{-1}$ is a bijection from W to the set of Borels of G containing T.

Definition 3.63. A linear algebraic k-group G is *quasi-split* if there exists a Borel k-subgroup B of G.

Exercise 3.64. If D is a finite-dimensional division algebra over k with center k and $D \neq k$, prove that $D^\times$ is not quasi-split.

Remark 3.65. It is a rather non-obvious fact (of course suggested by the terminology) that split groups are quasi-split. See Exercise 4.19(5)

3.5. **The dynamic method.** Throughout this section, let k be a field. The dynamic method is an almost magical machine which trivializes several difficult-looking results in the theory of algebraic groups.

Definition 3.66. Let R be a k-algebra, let X be a separated R-scheme, and let $f\colon (\mathbf{G}_\mathrm{m})_R \to X$ be an R-morphism. We say that *the limit* $\lim_{t\to 0} f(t)$ *exists* provided that there is an extension of f to an R-morphism $\overline{f}\colon \mathbf{A}^1_R \to X$. Write $\lim_{t\to 0} f(t) = x$ if $\overline{f}(0) = x$.

Exercise 3.67. Prove that in Definition 3.66, if f extends to an R-morphism $\mathbf{A}^1_R \to X$, then this extension is unique.[63] In particular, if $\lim_{t\to 0} f(t)$ exists, then there is a unique x such that $\lim_{t\to 0} f(t) = x$.

Definition 3.68. Let G be a linear algebraic k-group, and let $\lambda\colon \mathbf{G}_\mathrm{m} \to G$ be a k-homomorphism. Define subfunctors $P_G(\lambda)$, $Z_G(\lambda)$, and $U_G(\lambda)$ of G as follows: if R is a k-algebra, then

$$P_G(\lambda)(R) = \{g \in G(R)\colon \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\},$$

$$Z_G(\lambda)(R) = \{g \in G(R)\colon (\mathbf{G}_\mathrm{m})_R \text{ acts trivially on } g\},$$

and

$$U_G(\lambda)(R) = \{g \in G(R)\colon \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1\}.$$

Exercise 3.69. Show that $P_G(\lambda)$, $Z_G(\lambda)$, and $U_G(\lambda)$ are subgroup functors of G.

Exercise 3.70. This exercise shows that $P_G(\lambda)$, $Z_G(\lambda)$, and $U_G(\lambda)$ are all representable by closed k-subgroup schemes of G, and $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$.

(1) By Exercise 3.12, the action of $\mathbf{G}_\mathrm{m}$ on G through conjugation by λ induces a weight space decomposition
$$k[G] = \bigoplus_{n\in\mathbf{Z}} k[G]_n.$$
Show that $P_G(\lambda)$ is represented by the closed subscheme of G defined by the ideal generated by $\{k[G]_n\}_{n<0}$.
(2) Show that $Z_G(\lambda) = P_G(\lambda) \cap P_G(\lambda^{-1})$, and deduce that $Z_G(\lambda)$ is representable by a closed subgroup of $P_G(\lambda)$.

[62] *Hint*: Prove $B \subset \mathscr{D}(N_G(B))$ and use Exercise 3.6.

[63] *Hint*: if $\overline{f}_1$ and $\overline{f}_2$ are two extensions, consider the equalizer of $\overline{f}_1$ and $\overline{f}_2$, a subscheme of $\mathbf{A}^1_R$. Note that the valuative criterion of separatedness is not sufficient if R is not a field!

(3) Show that there is a k-homomorphism $P_G(\lambda) \to G$ defined by $g \mapsto \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$, and show that its image is $Z_G(\lambda)$.
(4) Deduce that $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$.

Exercise 3.71. It is a fact (see [Con22, Lemma 24.3.3]) that $P_G(\lambda)$, $Z_G(\lambda)$, and $U_G(\lambda)$ are all smooth, and the multiplication map $U_G(\lambda^{-1}) \times P_G(\lambda) \to G$ is an open embedding. Prove this when $\operatorname{char} k = 0$ using Remark 2.2 and a calculation of the tangent spaces of $U_G(\lambda^{-1})$ and $P_G(\lambda)$.

Exercise 3.72. Using Exercise 3.71, show that if G is a connected linear algebraic group and $T \subset G$ is a torus, then $Z_G(T)$ is connected.

4. Reductive groups

The theory of reductive groups is vast, and we can only give a limited introduction to the theory of root systems, forms, and integral models. Very good references for this material include [Con22], [Bou68], and [KP23].

4.1. Definitions and small examples.

The definition of reductive group is somewhat unintuitive at first. A first try (and the reason for the word) is as follows.

Definition 4.1. Say G is *linearly reductive* if for every field extension K/k, every K-linear representation of G_K is semisimple (i.e., completely reducible).

Exercise 4.2. Show that tori are linearly reductive.[64]

Exercise 4.3. (Easy) Show that the group of upper-triangular matrices in GL_n is not linearly reductive.

In characteristic 0, linear reductivity is the right notion, but in characteristic $p > 0$ it is a theorem of Nagata [Nag62] that tori are the only connected linear algebraic groups which are linearly reductive.

Exercise 4.4. (SL_2 is not linearly reductive in positive characteristic.) Let k be a field of characteristic $p > 0$, and let V be the standard representation of SL_2 over k.

(1) Show that $\operatorname{Sym}^p V \cong k[x,y]_{\deg = p}$, where SL_2 acts on $k[x,y]$ by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^i y^j = (ax+cy)^i (bx+dy)^j.$$
(2) Show that $\operatorname{Sym}^p V$ is not irreducible: in fact, $V^{(p)} := kx^p \oplus ky^p$ is SL_2-stable.
(3) Show that $V^{(p)}$ does not admit an SL_2-stable complement in $\operatorname{Sym}^p V$, and in particular $\operatorname{Sym}^p V$ is not semisimple.

Thus linear reductivity is a very stringent condition in positive characteristic. Motivated by this, we search for another definition.

Exercise 4.5. Let $\mathscr{R}_u(G_{\overline{k}})$ be the largest smooth connected normal unipotent $\overline{k}$-subgroup of $G_{\overline{k}}$. If k is perfect, show that there is a unique normal subgroup $\mathscr{R}_u(G) \subset G$ such that $\mathscr{R}_u(G)_{\overline{k}} = \mathscr{R}_u(G_{\overline{k}})$.[65] The group $\mathscr{R}_u(G)$ is called the *unipotent radical* of G.

Definition 4.6. A linear algebraic group G is *reductive* if $\mathscr{R}_u(G^0_{\overline{k}}) = 1$.

Exercise 4.7. Show that linearly reductive groups are reductive.[66]

[64] *Hint*: use Exercise 3.12.
[65] *Hint*: use Galois descent.
[66] *Hint*: use Exercise 3.52.

Exercise 4.8. Show that if N is a smooth connected closed k-subgroup of a reductive k-group G, then N is reductive.[67]

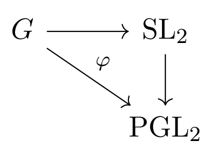
Exercise 4.9. Show that a solvable connected reductive group is a torus.[68]

Exercise 4.10. Show that SL_2 and PGL_2 are split connected reductive groups.[69]

In fact, SL_2 and PGL_2 are enormously special among reductive groups, as the following exercise illustrates.

Exercise 4.11. (Important: illustrates many general techniques from Section 3) This exercise shows that SL_2 and PGL_2 are the *only* non-commutative split connected reductive k-groups G of rank 1 over a field k.

(1) Reduce to the case that G is quasi-split (Definition 3.63).[70]
(2) Let $T \subset G$ be a split maximal k-torus and let $B \subset G$ be a Borel k-subgroup. Using Exercise 3.60, show that the set of Borel $\overline{k}$-subgroups of $G_{\overline{k}}$ containing $T_{\overline{k}}$ is in bijection with the set of fixed points for the action of $T_{\overline{k}}$ on $(G/B)_{\overline{k}}$.
(3) Deduce from Exercise 3.49 and Exercise 3.62 that $\dim(G/B) = 1$ and, if $W = (N_G(T)/Z_G(T))(\overline{k})$, then $|W| = 2$.
(4) Prove that $G/B \cong \mathbf{P}^1_k$.[71]
(5) Use (3) to obtain a k-homomorphism $\varphi\colon G \to \mathrm{PGL}_2$.
(6) Show that $\ker(\varphi)^0_{\mathrm{red}} \subset B$ and thus $N := \ker(\varphi)^0_{\mathrm{red}}$ is solvable. Deduce from Exercise 3.36 and non-solvability of G that φ is surjective.
(7) Using Exercises 4.9 and 3.13, show that N is a central torus of G. Deduce from Exercise 3.44 that N is trivial and thus φ is finite and $\dim G = 3$.
(8) Using 3.42, show that after passing to a $\mathrm{PGL}_2(k)$-conjugate of φ, we may assume that φ sends T to the diagonal split k-torus of PGL_2. Deduce that if $\lambda\colon \mathbf{G}_{\mathrm{m}} \cong T$ is an isomorphism, then $\dim U_G(\lambda^{\pm 1}) = 1$ (with notation as in Section 3.5).[72]
(9) Using Exercise 3.34, show that $U_G(\lambda^{\pm 1}) \cong \mathbf{G}_{\mathrm{a}}$, and $B^{\pm} = T \ltimes U_G(\lambda^{\pm 1})$ are two Borel k-subgroups with intersection T. Deduce that $\ker\varphi \subset T$ and thus $\ker\varphi \cong \mu_e$ for some $e \geq 1$ by Exercise 3.24.
(10) Using the already-proved fact that $|W| = 2$, show that $e|2$. We are done if $e = 1$.
(11) Show that if H and K are smooth connected k-groups and f is a rational map from H to K which respects multiplication (in the sense of rational maps from $H \times H$ to K), then f is a homomorphism.[73]
(12) Using Exercise 3.71, show that if $e = 2$ then there is an isomorphism $G \cong \mathrm{SL}_2$ such that the diagram



commutes, and conclude.

In the next section, we will see *many* examples in which results about general reductive groups are proven by appealing to a simple calculation in SL_2 or PGL_2.

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[67] *Hint*: the unipotent radical of N is preserved by all k-automorphisms.

[68] *Hint*: use Exercise 3.32.

[69] *Hint*: using properties of the diagonal torus, show that any nontrivial smooth connected normal unipotent subgroup of SL_2 must contain all of the unipotent elements of $\mathrm{SL}_2(\overline{k})$.

[70] *Hint*: assume that the result holds over $\overline{k}$ and use the dynamic method.

[71] *Hint*: it is enough to show that G/B is of genus 0.

[72] *Hint*: use Exercise 3.71.

[73] *Hint*: reduce to $k = k_s$ and use the fact that k_s-points are dense in smooth varieties.

4.2. **Roots.** Let G be a split linear algebraic group over a field k, and let T be a split maximal k-torus of G, as exists by Theorem 3.37. By Exercise 3.12, if $\mathfrak{g}$ is the Lie algebra of G (i.e., the tangent space of G at the identity), then the adjoint representation of T on $\mathfrak{g}$ (i.e., the representation coming from the conjugation action of $T(k)$ on $\mathfrak{g} = \ker(G(k[\epsilon]/(\epsilon^2)) \to G(k))$) splits up into

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the α-eigenspace for the action of T on $\mathfrak{g}$. A *root* of the pair (G, T) is a nonzero element α of $X^*(T)$ such that $\mathfrak{g}_\alpha \neq 0$. The set of roots is denoted by $\Phi(G, T)$.

Exercise 4.12. Compute the sets of roots for GL_n, SL_n, Sp_{2n}, and SO_n with respect (in each case) to the diagonal torus T. Show in each case that $\mathfrak{g}_0 = \mathrm{Lie}\, T$.

Exercise 4.13. Show that the natural action of the Weyl group (Exercise 3.53) on $X^*(T)$ preserves $\Phi(G, T)$.

The following theorem is *very* tricky, and it is hard to extract a moral from the proof. However, once it is proven, it will combine with Exercise 4.11 to prove almost everything else.

Theorem 4.14. [Con22, Theorem 2.1.3] *If G is a reductive group and $T \subset G$ is a torus, then $Z_G(T)$ is reductive.*

Exercise 4.15. Using Theorem 4.14, show that if T is a maximal k-torus of a connected reductive group G, then $Z_G(T) = T$.[74]

Exercise 4.16. Show that if $f\colon G \to G'$ is a surjection of linear algebraic groups and G is reductive, then G' is reductive.[75]

Definition 4.17. Recall that if G is a linear algebraic group over a field k, then one can define the *derived group* $\mathscr{D}G$ [Con20, Example 16.2.4]. This is a closed linear algebraic subgroup of G with the property that $(\mathscr{D}G)(\overline{k}) = [G(\overline{k}), G(\overline{k})]$, where $[\Gamma, \Lambda]$ denotes the group generated by the commutators $[\gamma, \lambda] \coloneqq \gamma\lambda\gamma^{-1}\lambda^{-1}$ for $\gamma \in \Gamma$ and $\lambda \in \Lambda$. Say that G is *perfect* if $\mathscr{D}G = G$.

Exercise 4.18. (Classical) Show that SL_2 and PGL_2 are perfect.

Exercise 4.19. Using Theorem 4.14 and Exercise 4.11, show that if G is a split connected reductive group, T is a split maximal k-subtorus of G, and α is a root of (G, T), then the derived group (Definition 4.17) of $Z_G((\ker\alpha)^0_{\mathrm{red}})$[76] is isomorphic to SL_2 or PGL_2. Deduce the following facts:

1. $\Phi(G, T)$ is closed under negation, i.e., if $\mathfrak{g}_\alpha \neq 0$ then $\mathfrak{g}_{-\alpha} \neq 0$.
2. If $\alpha \in \Phi(G, T)$ and $r\alpha \in \Phi(G, T)$ for some $r \in \mathbf{Q}$, then $r \in \{\pm 1\}$.
3. If $\alpha \in \Phi(G, T)$, then $\dim_k \mathfrak{g}_\alpha = 1$.
4. For each $\alpha \in \Phi(G, T)$, there is an element s_α of the Weyl group and cocharacter $\alpha^\vee\colon \mathbf{G}_{\mathrm{m}} \to T \cap \mathscr{D}G$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and $s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha$ for all $\beta \in X^*(T)$. The element $\alpha^\vee$ is called the *coroot* associated to α, and the set of coroots is denoted by $\Phi^\vee$.
5. The Borel $\overline{k}$-subgroups of $G_{\overline{k}}$ are precisely the subgroups of the form $P_{G_{\overline{k}}}(\lambda)$, where $\lambda\colon (\mathbf{G}_{\mathrm{m}})_{\overline{k}} \to T_{\overline{k}}$ is a cocharacter such that $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Phi(G, T)$.[77]
6. Any group of the form $P_G(\lambda)$ is parabolic.
7. Split linear algebraic groups are quasi-split.[78]

[74] *Hint*: apply Exercise 3.44 to the group $Z_G(T)/T$.
[75] *Hint*: use Exercise 4.8.
[76] It is a technical and non-obvious fact that $(\ker\alpha)^0_{\mathrm{red}}$ is a smooth group scheme even if k is imperfect.
[77] *Hint*: show that a linear algebraic group is solvable if and only if it has no non-commutative reductive quotients.
[78] *Hint*: reduce to the reductive case and use (4).

Remark 4.20. In fact, every parabolic k-subgroup of G is of the form $P_G(\lambda)$ for some cocharacter $\lambda\colon \mathbf{G}_{\mathrm{m}} \to G$ [Con22, Theorem 6.1.1]; this fact immediately implies the useful fact that $\mathscr{D}G$ is k-anisotropic (Definition 4.59) if and only if it does not contain any proper parabolic k-subgroups.

Proposition 4.21. [CGP15, Proposition 3.3.6] *Let G be a connected reductive k-group, and let $S \subset G$ be a split k-torus. If $A \subset X^*(S)$ is a subsemigroup, then there is a unique connected linear algebraic subgroup $H_A(G) \subset G$ normalized by S such that* $\mathrm{Lie}\, H_A(G) = \bigoplus_{a \in A \cap \Phi(G,S)} \mathfrak{g}_a$. *If $0 \notin A$, then $H_A(G)$ is unipotent and k-split solvable (in which case it is denoted by $U_A(G)$).*

Exercise 4.22. Prove that if G is a split connected reductive k-group, T is a split maximal k-torus in G, and $\alpha \in \Phi(G,T)$, then $U_\alpha \coloneqq U_{\{\alpha\}}(G)$ is 1-dimensional. The group U_α is called the *root group* corresponding to α. Compute the root groups for SL_n and Sp_4.

Definition 4.23. A linear algebraic group G over a field k is *semisimple* if G is reductive and does not contain any nontrivial central k-torus.

Exercise 4.24. Prove that a connected semisimple k-group G is perfect, i.e., satisfies $\mathscr{D}G = G$.[79]

If G is merely connected reductive, prove that $\mathscr{D}G$ is semisimple. Moreover, if Z is the maximal central k-torus in G, prove that the multiplication map $Z \times \mathscr{D}G \to G$ is a central isogeny (Definition 4.35).

Definition 4.25. A *root datum* is a quadruple $(X, \Phi, Y, \Phi^\vee)$, where

(1) X and Y are finite free $\mathbf{Z}$-modules equipped with a perfect pairing $X \otimes_{\mathbf{Z}} Y \to \mathbf{Z}$,
(2) $\Phi \subset X - \{0\}$ and $\Phi^\vee \subset Y - \{0\}$ are finite subsets equipped with a bijection $\Phi \to \Phi^\vee$, $\alpha \mapsto \alpha^\vee$, such that $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$,
(3) for each $\alpha \in \Phi$, the maps $s_\alpha\colon X \to X$ and $s_{\alpha^\vee}\colon Y \to Y$ defined by $s_\alpha(x) \coloneqq x - \langle \alpha^\vee, x \rangle \alpha$ and $s_{\alpha^\vee}(y) \coloneqq y - \langle \alpha, y \rangle \alpha^\vee$ preserve Φ and $\Phi^\vee$, respectively.

The *Weyl group* $W(\Phi)$ of the root datum is the subgroup of $\mathrm{Aut}_{\mathbf{Z}}(X)$ generated by the reflections s_α. Say that $(X, \Phi, Y, \Phi^\vee)$ is *irreducible* if the representation of $W(\Phi)$ on $X \otimes_{\mathbf{Z}} \mathbf{C}$ is irreducible.

Exercise 4.26. Let G be a split connected reductive k-group, and let $T \subset G$ be a split maximal k-torus. Show that $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is a root datum.[80] Prove that if it is irreducible, then G is semisimple.

Exercise 4.27. Verify that the Weyl group of the root datum for SL_n is isomorphic to the Weyl group of SL_n (i.e., S_n). In fact, this is true for the root datum of *every* connected reductive group.

Exercise 4.28. Show that if $(X, \Phi, Y, \Phi^\vee)$ is a root datum, then $(Y, \Phi^\vee, X, \Phi)$ is also a root datum. (This accounts for the symmetry in Definition 4.25.)

Definition 4.29. A *basis* of a root datum $(X, \Phi, Y, \Phi^\vee)$ is a collection $a_1, \dots, a_n \in \Phi$ of vectors forming a basis of $X \otimes_{\mathbf{Z}} \mathbf{R}$ such that every element of Φ can be written in the form $\sum_{i=1}^n c_i a_i$ for elements $c_i \in \mathbf{Z}$ such that either every c_i is a non-negative integer or every c_i is a non-positive integer.

A *system of positive roots* is a subset $\Phi^+ \subset \Phi$ such that $\Phi = \Phi^+ \sqcup -\Phi^+$ and Φ^+ is closed under addition in Φ (i.e., if $a, b \in \Phi^+$ are such that $a + b \in \Phi$, then $a + b \in \Phi^+$.

Exercise 4.30. Show that if $a_1, \dots, a_n \in \Phi$ is a basis, then the set Φ^+ of roots which are $\mathbf{Z}_{\geq 0}$-linear combinations of elements of $a_1, \dots, a_n$ is a system of positive roots.

Conversely, if Φ^+ is a system of positive roots, let Δ_{Φ^+} denote the set of roots in Φ^+ which cannot be written as the sum of two elements of Φ^+. It is a fact that Δ_{Φ^+} is a basis [Bou68, Chapter VI, §§1.5-1.7].

[79] *Hint*: what happens for SL_2 and PGL_2?
[80] *Hint*: Exercise 4.19

Exercise 4.31. Let $(X, \Phi, Y, \Phi^\vee)$ be the root datum corresponding to (SL_n, T), where T is the diagonal torus of SL_n. Show that the characters $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_i t_{i+1}^{-1}$ $(1 \leq i \leq n-1)$ form a basis for the root datum. Describe the corresponding system of positive roots.

Definition 4.32. Let $\Delta \subset \Phi$ be a basis for the root datum $(X, \Phi, Y, \Phi^\vee)$, and choose a $W(\Phi)$-stable inner product on $X \otimes_{\mathbf{Z}} \mathbf{R}$. The *Dynkin diagram* is the weighted directed graph with vertex set Δ, where an edge connects $a, b \in \Delta$ if and only if $(a, b) \neq 0$. Furthermore, if $||a|| \geq ||b||$ then the edge connecting a and b points toward b and has weight $\frac{||a||^2}{||b||^2}$.

Exercise 4.33. (Unimportant) Show that the Dynkin diagram of a root datum is independent of the choice of basis and the choice of $W(\Phi)$-stable inner product. Furthermore, the root datum is irreducible if and only if the Dynkin diagram is connected.

Exercise 4.34. (Important) Compute the Dynkin diagrams associated to SL_n, Sp_{2n}, and SO_n. (For SO_n, split into cases according to whether n is even or odd.)

Definition 4.35. A homomorphism $f : G \to H$ of connected reductive groups over k is a *central isogeny* if it is surjective and has finite central kernel.

Exercise 4.36. Show that the natural map $\mathrm{SL}_n \to \mathrm{PGL}_n$ is a central isogeny.

Exercise 4.37. Let $f : G \to H$ be a central isogeny of split connected reductive groups.

1. Show that f maps a split maximal k-torus of G to a split maximal k-torus of H.
2. If S is a maximal k-torus of G and $T = f(S)$, then the induced map $X^*(T) \to X^*(S)$ maps $\Phi(H, T)$ to $\Phi(G, S)$. Similarly, the map $X_*(S) \to X_*(T)$ maps $\Phi^\vee(G, S)$ bijectively to $\Phi^\vee(H, T)$.
3. Deduce that if $X_*(T) = \mathbf{Z}\Phi^\vee(H, T)$ or $X^*(S) = \mathbf{Z}\Phi(G, S)$, then f is an isomorphism. In the former case, H is called *simply connected*; in the latter case, G is said to be *of adjoint type*.
4. Show that SL_n and Sp_{2n} are both simply connected, whereas PGL_n and SO_{2n+1} are of adjoint type. For SO_{2n}, what can you say about $X_*(T)/\mathbf{Z}\Phi^\vee$ and $X^*(T)/\mathbf{Z}\Phi$?

Theorem 4.38 (Existence and Isomorphism Theorems)**.** *Let k be a field. The morphism $(G, T, M) \mapsto (X^*(T), \Phi(G, T), X_*(T), \Phi^\vee(G, T))$ is an equivalence of categories*

$$\{\textit{triples } (G, T, M)\} \to \{\textit{root data } (X, \Phi, Y, \Phi^\vee)\},$$

where (G, T, M) is a triple of a split connected reductive group G, a split maximal k-torus $T \subset G$, and an isomorphism $M : X^*(T) \cong \mathbf{Z}^r$. Morphisms on the left are given by central isogenies compatible with the identifications of character lattices, and morphisms on the right are given by homomorphisms between the corresponding X and Y which respect the chosen perfect pairings and preserve the sets Φ and $\Phi^\vee$.

Exercise 4.39. Deduce from Theorem 4.38 that for every semisimple k-group G, there is a unique-up-to-isomorphism central isogeny $\pi : \widetilde{G} \to G$ such that $\widetilde{G}$ is simply connected.

Exercise 4.40. By Theorem 4.38 and Exercise 4.28, if (G, T) is a pair consisting of a split connected reductive k-group G and a k-split maximal k-torus $T \subset G$, then there is a corresponding pair $(G^\vee, T^\vee)$ with

$$(X^*(T^\vee), \Phi(G^\vee, T^\vee), X_*(T^\vee), \Phi^\vee(G^\vee, T^\vee)) \cong (X_*(T), \Phi^\vee(G, T), X^*(T), \Phi(G, T)).$$

The group $G^\vee$ is called the *Langlands dual group* of G. Note that $(G^\vee)^\vee \cong G$. Prove the following isomorphisms:

1. $\mathrm{GL}_n^\vee \cong \mathrm{GL}_n$,
2. $\mathrm{SL}_n^\vee \cong \mathrm{PGL}_n$,
3. $\mathrm{Sp}_{2n}^\vee \cong \mathrm{SO}_{2n+1}$,
4. $\mathrm{SO}_{2n}^\vee \cong \mathrm{SO}_{2n}$.

4.3. **Forms and Galois cohomology.** The canonical reference for everything in this section is [Ser02]. We will assume familiarity with the usual notions of group cohomology with coefficients in an abelian group, but we will briefly develop the (significantly more limited) non-abelian case.

Let k be a field with absolute Galois group Γ, and let X be a set with an action of Γ. We define $\mathrm{H}^0(\Gamma, X) = X^\Gamma$, the set of fixed points for the action of Γ on X. If X is a group and Γ acts through group automorphisms, we further define $\mathrm{H}^1(\Gamma, X)$ as follows (see [Ser02, Appendix to Chapter 1]): say that a function $a\colon \Gamma \to X$ is a *1-cocycle* if it is continuous[81] and satisfies $a(\gamma\delta) = a(\gamma) \cdot {}^\gamma a(\delta)$ for all $\gamma, \delta \in \Gamma$. Say that two 1-cocycles a and a' are *cohomologous* if there exists $x \in X$ such that $a'(\gamma) = x^{-1} \cdot a(\gamma) \cdot {}^\gamma x$ for all $\gamma \in \Gamma$.

Exercise 4.41. (Easy) Show that the relation "$a \sim a'$ if and only if a and a' are cohomologous" is an equivalence relation.

By definition, $\mathrm{H}^1(\Gamma, X)$ is the quotient of the set of 1-cocycles $\Gamma \to X$ by the equivalence relation of being cohomologous. Note that $\mathrm{H}^1(\Gamma, X)$ is a pointed set (with distinguished point corresponding to the cocycle $a(\gamma) = 1$), and if X is abelian then it is a group which agrees with the usual group cohomology. However, if X is non-abelian then $\mathrm{H}^1(\Gamma, X)$ has no natural group structure in general. On the other hand, $\mathrm{H}^0(\Gamma, X)$ is of course a group.

Exercise 4.42. [Ser02, Chapter 1, §5] (If you haven't checked something like this before, it is worth doing it once.) Show that $\mathrm{H}^0(\Gamma, X)$ and $\mathrm{H}^1(\Gamma, X)$ are both functorial in X.

(1) If $X \subset Y$ is an inclusion of groups with Γ-action, then there is an exact sequence
$$1 \to \mathrm{H}^0(\Gamma, X) \to \mathrm{H}^0(\Gamma, Y) \to \mathrm{H}^0(\Gamma, Y/X) \xrightarrow{\delta} \mathrm{H}^1(\Gamma, X) \to \mathrm{H}^1(\Gamma, Y) \tag{1}$$
in which δ is defined as follows: given $z \in (Y/X)^\Gamma$, choose a lift $y \in Y$ of z and define $a(\gamma) = y^{-1} \cdot {}^\gamma y \in X$.

(2) If X is normal in Y, show that (1) extends to an exact sequence
$$1 \to \mathrm{H}^0(\Gamma, X) \to \mathrm{H}^0(\Gamma, Y) \to \mathrm{H}^0(\Gamma, Y/X) \xrightarrow{\delta} \mathrm{H}^1(\Gamma, X) \to \mathrm{H}^1(\Gamma, Y) \to \mathrm{H}^1(\Gamma, Y/X). \tag{2}$$

(3) If X is central in Y, show that (2) extends to an exact sequence
$$1 \to \mathrm{H}^0(\Gamma, X) \to \mathrm{H}^0(\Gamma, Y) \to \mathrm{H}^0(\Gamma, Y/X) \xrightarrow{\delta} \mathrm{H}^1(\Gamma, X) \to \mathrm{H}^1(\Gamma, Y) \to \mathrm{H}^1(\Gamma, Y/X) \xrightarrow{\Delta} \mathrm{H}^2(\Gamma, X),$$
where Δ is defined in an analogous way to δ. (See [Ser02, Chapter 1, §5.6].)

Definition 4.43. A *k-form* of G is an algebraic k-group H such that $H_{\overline{k}} \cong G_{\overline{k}}$.

Exercise 4.44. Show that if Γ is the absolute Galois group of k and G is a connected reductive k-group, then the set of k-forms of G is in natural bijection with $\mathrm{H}^1(\Gamma, \mathrm{Aut}_{k_s}(G_{k_s}))$.

Definition 4.45. An *inner form* of G is a k-form H of G such that there is a k_s-isomorphism $\alpha\colon H_{k_s} \cong G_{k_s}$ such that ${}^\sigma\alpha \circ \alpha^{-1}\colon G_{k_s} \to G_{k_s}$ is an inner automorphism for all $\sigma \in \mathrm{Gal}(k_s/k)$. An *outer form* of G is a k-form which is not an inner form.

Exercise 4.46. Show that the set of inner forms of G is in canonical bijection with $\mathrm{H}^1(\Gamma, G^{\mathrm{ad}}(k_s))$, where $G^{\mathrm{ad}} = G/Z(G)$ is the so-called adjoint group of G.

Exercise 4.47. If (V, B) is as in Exercise 2.7, show that $\mathrm{SO}(V, B)$ is an inner form of $\mathrm{SO}_{\dim V}$. Show that $A^\times$ is an inner form of GL_n, where $\dim_k A = n^2$, and $A^{N=1}$ is an inner form of SL_n. Moreover, show that $\mathrm{U}(h)$ is an *outer* form of GL_n, where $n = \dim_{k'} V$.

Exercise 4.48. Let k be a finite extension of $\mathbf{Q}_p$ for some $p > 0$.

(1) Let $n \geq 1$ be an integer, and compute $\mathrm{H}^i(k, \mu_n)$ for $i \geq 0$. Notice in particular that these groups are *finite*.

[81] Endow X with the discrete topology.

(2) Using the Hochschild–Serre spectral sequence, deduce that $\mathrm{H}^i(k, M)$ is finite for every finite Galois module M.

4.4. **Integral models and bounded subgroups.** Throughout this section, let (k, ω) be a henselian discretely valued field with perfect residue field $\mathfrak{f}$ and ring of integers $\mathfrak{o}$, and let G be a connected reductive group over k.

Definition 4.49. If X is a finite type affine k-scheme, an *integral model* of X is a pair $(\mathscr{X}, \alpha)$, where $\mathscr{X}$ is a flat affine $\mathfrak{o}$-scheme and $\alpha\colon \mathscr{X}_k \to X$ is a k-isomorphism. If $X = G$, then we require furthermore that $\mathscr{X}$ is an $\mathfrak{o}$-group scheme and α is an isomorphism of k-groups.

Note that α induces an inclusion $\mathscr{X}(\mathfrak{o}) \subset X(k)$. When talking about integral models, we will often omit α, treating it as implicit.

Exercise 4.50. Let X be a finite type affine k-scheme, and let $\mathscr{X}$ be an integral model of X. Show that $\mathfrak{o}[\mathscr{X}] \otimes_{\mathfrak{o}} k = k[X]$.

A large part of Bruhat–Tits theory concerns the existence of interesting integral models of G. Because it concerns schemes over discrete valuation rings, this section requires considerably more experience with algebraic geometry than the other sections. The following exercise is a first step to building integral models.

Exercise 4.51. Let $\mathscr{X}$ be an affine $\mathfrak{o}$-scheme, and let $Z \subset \mathscr{X}_k$ be a closed subscheme. Show that the schematic closure $\mathscr{Z}$ of Z in $\mathscr{X}$ is flat over $\mathfrak{o}$.[82]

Remark 4.52. It is a difficult theorem of Néron (see [BLR90, §7.1, Theorem 5] and Raynaud's theorem [PY06, Proposition 3.1]) that if $\mathscr{G}$ is an integral model of G then there exists a homomorphism $\mathscr{G}' \to \mathscr{G}$ of integral models such that $\mathscr{G}'(\mathcal{O}) = \mathscr{G}(\mathcal{O})$ for any étale extension $\mathcal{O}$ of $\mathfrak{o}$.

Definition 4.53. If X is a finite type affine k-scheme, a subset $B \subset X(k)$ is *bounded* if for every $f\colon X \to \mathbf{A}^1$, the subset $f(B) \subset k$ is bounded in the valuation topology.

Exercise 4.54. (Basic properties of boundedness) Let X, Y, and Z be finite type affine k-schemes, and let $B \subset X(k)$ be a bounded subset.

(1) If k is locally compact (so $\mathfrak{f}$ is finite), show that B has compact closure (in the analytic topology).
(2) If B is bounded and $f\colon X \to Y$ is a k-morphism, show that $f(B) \subset Y(k)$ is bounded.
(3) If $f\colon Z \to X$ is a finite k-morphism, show that $f^{-1}(B) \subset Z(k)$ is bounded.
(4) If $C \subset (X \times Y)(k)$ is a subset, then it is bounded if and only if its projections to $X(k)$ and $Y(k)$ are both bounded.
(5) If $\mathscr{X}$ is an integral model of X, then $\mathscr{X}(\mathfrak{o}) \subset X(k)$ is bounded.

Remark 4.55. If $\mathscr{X}$ is an integral model of X, then the subset $\mathscr{X}(\mathfrak{o}) \subset X(k)$ is called *schematic*. A reasonable question is whether $\mathscr{X}(\mathfrak{o})$ *determines* $\mathscr{X}$, i.e., if $\mathscr{X}$ and $\mathscr{X}'$ are integral models of X such that $\mathscr{X}(\mathfrak{o}) = \mathscr{X}'(\mathfrak{o})$, is it the case that $\mathscr{X} \cong \mathscr{X}'$? By Remark 4.52, this question is only reasonable when $\mathscr{X}$ and $\mathscr{X}'$ are smooth, and in this case [KP23, Corollary 2.10.10] shows the answer is yes provided that $\mathfrak{f}$ is separably closed.

Exercise 4.56. If $\Gamma \subset G(k)$ is a maximal bounded subgroup, show that there is an integral model $\mathscr{G}$ of G such that $\Gamma = \mathscr{G}(\mathfrak{o})$.[83]

Exercise 4.57. (Prasad) Let $\Gamma \subset \mathrm{GL}_n(k)$ be an unbounded subgroup. The first three parts of this exercise will show by way of contradiction that there exists $\gamma \in \Gamma$ such that $\gamma^{\mathbf{Z}}$ is unbounded.

[82] *Hint*: by definition, if $\mathscr{X} = \operatorname{Spec} A$ and $Z = \operatorname{Spec}(A \otimes_{\mathfrak{o}} k)/I$, then $\mathscr{Z} = \operatorname{Spec} A/J$, where J is the preimage of I under the map $A \to A \otimes_{\mathfrak{o}} k$.

[83] *Hint*: choose an embedding $G \subset \mathrm{GL}_n$ and use Exercise 4.51.

(1) Reduce to the case that k^n is an irreducible representation of Γ.
(2) (Burnside) Show that Γ spans $\mathrm{M}_n(k)$ as a k-algebra.[84]
(3) Let $A = \sum_\alpha \mathfrak{o}\gamma_\alpha$ be the finite $\mathfrak{o}$-submodule of $\mathrm{M}_n(k)$ generated by a finite subset $\{\gamma_\alpha\}$ of Γ spanning $\mathrm{M}_n(k)$. Under the assumption that $\gamma^{\mathbf{Z}}$ is bounded for each $\gamma \in \Gamma$, use the trace pairing to deduce that Γ is contained in a finite $\mathfrak{o}$-submodule of $\mathrm{M}_n(k)$, contradicting unboundedness.
(4) Deduce that if $\Lambda \subset G(k)$ is a *Zariski-dense* bounded subgroup, then there is some $\lambda \in \Lambda$ such that $\lambda^{\mathbf{Z}}$ is unbounded.

Exercise 4.57 is remarkably consequential. We illustrate this with two examples.

Exercise 4.58. Let $\Gamma \subset G(k)$ be a bounded subgroup.
(1) Show that Γ is contained in an *open* bounded subgroup of $G(k)$.[85]
(2) Using Exercise 4.57, deduce that Γ is contained in a *maximal* bounded subgroup.

Definition 4.59. We say that G is (k-)*anisotropic* if it does not contain a nontrivial split k-torus.

Exercise 4.60. This exercise shows that $G(k)$ is bounded if and only if G is anisotropic.
(1) (Easy) Prove the forward direction.
(2) For the reverse direction, suppose first that $G = T$ is a torus. Show that if $T(k)$ is unbounded then there is a finite Galois extension k'/k, a k'-character $\chi\colon T_{k'} \to (\mathbf{G}_\mathrm{m})_{k'}$, and an element $t \in T(k)$ such that $\chi(t)$ has positive valuation.
(3) Let $\psi = \prod_{\sigma \in \mathrm{Gal}(k'/k)} \sigma^*(\chi)\colon T \to \mathbf{G}_\mathrm{m}$. Show that $\psi(t) \neq 1$, and deduce that T is isotropic.
(4) In the general case, show that if $G(k)$ is unbounded then there is a *semisimple* element $g \in G(k)$ such that $g^{\mathbf{Z}}$ is unbounded.
(5) Applying Theorem 3.37 to $Z_G(g)^0$, reduce to the already-resolved case that G is a torus.

Exercise 4.61. Let k be a finite extension of $\mathbf{Q}_p$. Show that a k-torus T is k-anisotropic if and only if $\mathrm{H}^2(k, T) = 0$.[86]

Exercise 4.62. Let $G(k)^1 = \{g \in G(k)\colon \omega(\chi(g)) = 0 \text{ for all } \chi\colon G \to \mathbf{G}_\mathrm{m}\}$.
(1) Show that every bounded subgroup of $G(k)$ is contained in $G(k)^1$.
(2) If $G = T$ is a torus, show that $T(k)^1$ is the unique maximal bounded subgroup of $T(k)$.[87]

5. Buildings

In this section we describe a very small part of the theory of buildings. Excellent references to go deeper into the ideas of this section include [KP23], [Tit79], [Yu09], and Joe Rabinoff's senior thesis.

5.1. The spherical building of a reductive group.

These exercises build on those of Section 4. To some extent, they are intended as a warm-up for Section 5.2.

Let G be a connected reductive group over a field k, and let $\mathcal{P}_G$ denote the set of proper parabolic k-subgroups of G, and let $\mathcal{P}_G^{\max}$ denote the set of *maximal* proper parabolic k-subgroups of G.

Definition 5.1. The *spherical building* of G is the pair $\mathcal{S}_G = (\mathcal{P}_G^{\max}, \mathcal{F}_G)$, where $\mathcal{F}_G$ consists of subsets $\{P_0, \dots, P_m\}$ of $\mathcal{P}_G^{\max}$ such that $\bigcap_{i=0}^m P_i \in \mathcal{P}_G$.

Exercise 5.2. Show that $\mathcal{S}_G$ is a simplicial complex of pure dimension $\operatorname{rk} G - 1$.

Exercise 5.3. Show that $\mathcal{S}_{\mathrm{GL}_n} \cong \mathcal{S}_n$, where $\mathcal{S}_n$ is defined as in Definition 2.39.

[84] *Hint*: either use the Jacobson density theorem or simply consult [Lam98].
[85] *Hint*: Exercise 2.30.
[86] *Hint*: use Tate–Nakayama duality.
[87] *Hint*: use the ideas in Exercise 4.60(2) and (3).

Exercise 5.4. Both $\mathcal{P}_G$ and the set of facets $\mathcal{F}_G$ of $\mathcal{S}_G$ are partially ordered sets (under inclusion). Show that the natural map $\mathcal{F}_G \to \mathcal{P}_G$ given by $\{P_0, \ldots, P_m\} \to \bigcap_{i=0}^m P_i$ is an order-reversing *bijection.*

Buildings are so-named because they are built out of apartments. If S is a maximal split k-torus of G, let $\mathcal{A}(S)$ be the full subcomplex of $\mathcal{S}_G$ consisting of maximal parabolic k-subgroups of G containing T, called the *apartment* corresponding to T.

Exercise 5.5. If T is the split maximal k-torus of GL_n corresponding to the basis B (i.e., T is the torus which stabilizes the lines spanned by elements of B), then $\mathcal{A}(T) = \mathcal{A}(B)$, with notation as in Exercise 2.41.

5.2. **Bruhat–Tits buildings.** The Bruhat–Tits building is somewhat complicated to define, and there is more than one way of doing so. We will focus on understanding the apartments of the Bruhat–Tits building.

Definition 5.6. A *root system* is a pair (V, Φ), where V is a finite-dimensional $\mathbf{R}$-vector space and $\Phi \subset V - \{0\}$ is a finite subset with the following properties.

(1) Φ spans V.
(2) for each $a \in \Phi$, there exists $a^\vee \in \Phi^\vee$ such that $\langle a, a^\vee \rangle = 2$ and $\langle b, a^\vee \rangle \in \mathbf{Z}$ for all $b \in \Phi$, and such that the reflection $r_a \colon V \to V$ defined by $r_a(b) = b - \langle b, a^\vee \rangle a$ preserves Φ.

The *rank* of (V, Φ) is $\dim_{\mathbf{R}} V$.

Remark 5.7. Roughly, root systems classify split connected semisimple groups up to central isogeny, while root data classify them up to isomorphism.

Definition 5.8. If V is a finite-dimensional $\mathbf{R}$-vector space, then an *affine space for* V is a set A equipped with a simply transitive action of V.[88] An *affine functional* is a function $\psi \colon A \to \mathbf{R}$ such that, for any fixed $a \in A$, the map $\dot{\psi}_a \colon V \to \mathbf{R}$ defined by $\dot{\psi}_a(v) := \psi(a + v) - \psi(a)$ is $\mathbf{R}$-linear. The set of affine functionals on A is denoted by A^*.

Exercise 5.9. If A is an affine space for V and $\psi \colon A \to \mathbf{R}$ is an affine functional, show that $\dot{\psi}_a$ is independent of the choice of $a \in A$. (Thus we will write $\dot{\psi}$ in place of $\dot{\psi}_a$ in the future.) Moreover, show that there is a short exact sequence

$$0 \to \mathbf{R} \to A^* \xrightarrow{\nabla} V^* \to 0,$$

where $\mathbf{R} \to A^*$ sends a real number r to the constant map $a \mapsto r$ and ∇ sends ψ to $\dot{\psi}$.

Definition 5.10. An *affine root system* is a pair (A, Ψ), where A is an affine space for a finite-dimensional $\mathbf{R}$-vector space V and $\Psi \subset A^* - \{0\}$ is a set of non-constant functionals such that

(1) Ψ spans A^*.
(2) $\Phi := \nabla(\Psi) \subset V^* - \{0\}$ is finite.
(3) for all $\psi \in \Psi$, there exists $\dot{\psi}^\vee \in V$ such that $\langle \psi, \dot{\psi}^\vee \rangle = 2$ and the reflection $r_{\psi, \dot{\psi}^\vee} \colon A^* \to A^*$ defined by $r_{\psi, \dot{\psi}^\vee}(\eta) = \eta - \langle \eta, \dot{\psi}^\vee \rangle \psi$ preserves Ψ.
(4) $\langle \dot{\eta}, \dot{\psi} \rangle \in \mathbf{Z}$ whenever $\eta, \psi \in \Psi$.
(5) for all $a \in \Phi$, the set $\{\psi \in \Psi \colon \dot{\psi} = a\}$ has at least two members and no limit point.

Say that (A, Ψ) is *reduced* if $\Psi \cap \mathbf{R}\psi = \{\pm\psi\}$ for all $\psi \in \Psi$. If $\psi \in \Psi$, then the set $H_\psi := \{a \in A \colon \psi(a) = 0\}$ is called the *root hyperplane* associated to ψ. A *chamber* of (A, Ψ) is a connected component $\mathcal{C}$ of $A - \bigcup_{\psi \in \Psi} H_\psi$, and a *wall* of $\mathcal{C}$ is a root hyperplane H_ψ intersecting the closure $\overline{\mathcal{C}}$. A *vertex* of $\mathcal{C}$ is a point of $\mathcal{C}$ lying in all but one wall of $\mathcal{C}$. A *facet* of (A, Ψ) is a nonempty subset of A of the form $\bigcap_{\psi \in S} H_\psi$, where S is a subset of Ψ.

[88] Essentially, A is "V without a choice of origin".

Exercise 5.11. Show that if (A, Ψ) is an affine root system, then $(V^*, \nabla(\Psi))$ is a root system.

Conversely, if (V^*, Φ) is a root system, consider $A = V$ as an affine space for itself and define $\Psi \subset A^* - \{0\}$ by

$$\Psi = \{a + n \colon a \in \Phi, n \in I_a\},$$

where $I_a = \mathbf{Z}$ if $a \notin 2\Phi$ and $I_a = 2\mathbf{Z} + 1$ if $a \in 2\Phi$. Show that (A, Ψ) is an affine root system, and draw the root hyperplanes in every case that Φ is irreducible of rank 2, labeling the walls of your favorite chamber (they are all $W(\Psi)$-conjugate).

Definition 5.12. If $x \in A$, let Ψ_x denote the set of $\psi \in \Psi$ such that $\psi(x) = 0$. Say that x is *special* if $\nabla \colon \mathbf{Q}\Psi_x \to \mathbf{Q}\Phi$ is surjective. Say that x is *extra special* if there exist $\psi_1, \dots, \psi_n \in \Psi$ such that $\psi_i(x) = 0$ for all i and such that $\{\dot{\psi}_1, \dots, \dot{\psi}_n\}$ is a basis for Φ.

Exercise 5.13. Prove that extra special points exist in A, and they are vertices. Moreover, show:

(1) In the affine root system associated to SL_n, every vertex is extra special.
(2) In the affine root system associated to Sp_4, there exist non-special vertices.[89]

For simplicity, let G be a split connected semisimple k-group. The Bruhat–Tits building is a simplicial complex $\mathcal{B}(G)$ with the following properties (among many others; see [KP23, §4]):

(1) There is an action of $G(k)$ on $\mathcal{B}(G)$ through simplicial automorphisms.
(2) If $G \to G'$ is a central isogeny, then the map $\mathscr{B}(G) \to \mathscr{B}(G')$ is an equivariant isomorphism.
(3) $\mathcal{B}(G) = \bigcup_S \mathcal{A}(S)$, where S ranges over the split maximal k-tori of G, and $\mathcal{A}(S)$ is the "apartment associated to S", isomorphic to the simplicial complex induced by the affine root system (A, Ψ) as in Exercise 5.11.
(4) The stabilizer of $\mathcal{A}(S)$ in $G(k)$ is $N_G(S)(k)$.
(5) Each $|\mathcal{A}(S)|$ is canonically an affine space for $X_*(S) \otimes_{\mathbf{Z}} \mathbf{R}$, and the action of $N_G(S)(k)$ on $|\mathcal{A}(S)|$ is through affine automorphisms; $S(k)$ acts by translations.
(6) There is a $G(k)$-invariant metric d on $|\mathcal{B}(G)|$ which restricts to a metric on $|\mathcal{A}(S)|$ which is induced by a Weyl-invariant metric on $X_*(S) \otimes_{\mathbf{Z}} \mathbf{R}$. Furthermore, d is non-positively curved in the sense that for all $x, y, z \in |\mathcal{B}(G)|$ there exists $m \in |\mathcal{B}(G)|$ such that

$$d(x,z)^2 + d(y,z)^2 \geq 2d(m,z)^2 + \frac{1}{2}d(x,y)^2.$$

Between any two points of $|\mathcal{B}(G)|$, there is a unique geodesic.
(7) If $x, y \in |\mathcal{B}(G)|$, then there is some apartment $\mathcal{A}(S)$ containing x and y.
(8) If $\mathcal{F}$ is a facet of $\mathcal{B}(G)$, then the set-wise stabilizer $G(k)_{\mathcal{F}}$ of $\mathcal{F}$ in $G(k)$ is bounded, and the set of fixed points for the action of $G(k)_{\mathcal{F}}$ on $|\mathcal{B}(G)|$ is contained in $|\mathcal{F}|$. If G is simply connected, then $G(k)_{\mathcal{F}}$ stabilizes $|\mathcal{F}|$ *pointwise*, not just setwise.
(9) There is a canonical smooth affine integral model $\mathscr{G}_{\mathcal{F}}$ of G (called the *parahoric group scheme* associated to $\mathcal{F}$) such that $\mathscr{G}_{\mathcal{F}}(\mathfrak{o}) = G(k)_{\mathcal{F}}$.[90] Furthermore, if $\mathcal{F} \subset \mathcal{A}(S)$ then S extends to a maximal $\mathfrak{o}$-torus $\mathcal{S}$ of $\mathscr{G}_{\mathcal{F}}$, and the root system of the maximal reductive quotient of $(\mathscr{G}_{\mathcal{F}})_{\mathfrak{f}}$ with respect to $\mathcal{S}_{\mathfrak{f}}$ is $\Psi_{\mathcal{F}} = \{\psi \in \Psi \colon \psi|_{\mathcal{F}} = 0\}$.
(10) If k'/k is an unramified field extension, then $\mathrm{Gal}(k'/k)$ acts on $\mathcal{B}(G_{k'})$ and $\mathcal{B}(G) = \mathcal{B}(G_{k'})^{\mathrm{Gal}(k'/k)}$.

Exercise 5.14. Verify as many of the above properties in the case $G = \mathrm{SL}_n$ as you can (or wish to). Sections 2.3 and 4.4 may be useful!

[89] *Hint*: draw a picture!

[90] Note in particular that this implies that if $G(k)$ stabilizes a facet, it must stabilize all of its points as well. By Exercise 2.50, this is rather special to the case that G is semisimple and simply connected.

Exercise 5.15. (Bruhat–Tits fixed point lemma) Let Γ be a group acting on $|\mathcal{B}(G)|$ such that $\Gamma \cdot x$ is bounded (with respect to the above metric) for every point $x \in |\mathcal{B}(G)|$. Prove that Γ has a fixed point.[91]

Deduce that if G is simply connected then the maximal bounded subgroups of $G(k)$ are *precisely* the stabilizers of vertices of $|\mathcal{B}(G)|$, and there are precisely $\operatorname{rk} G$ conjugacy classes of these.

The next few exercises use the axioms to establish some important results in Galois cohomology. In the following, let k be a local field, so k is complete and $\mathfrak{f}$ is finite.

Exercise 5.16. Let G be a split simply connected semisimple k-group. This exercise shows that $\mathrm{H}^1(k, G) = 1$.

(1) Let k^{un} be the maximal unramified extension of k, and use Steinberg's theorem [Ste65, Theorem 1.9][92] and Lang's theorem that k^{un} is C_1 [Lan52, Theorem 12] to reduce to proving that $\mathrm{H}^1(k^{\mathrm{un}}/k, G) = 1$.
(2) Choose a chamber $\mathcal{C} \subset \mathcal{A}(S)$ (for some split maximal torus S), and let $\mathcal{I} = G(k)_{\mathcal{F}}$ be the corresponding stabilizer (an *Iwahori subgroup*). Using the axioms, show that $\mathrm{H}^1(k^{\mathrm{un}}/k, \mathcal{I}) \to \mathrm{H}^1(k^{\mathrm{un}}/k, G)$ is surjective.[93]
(3) Using Exercise 3.38, use successive approximation to show that if $\mathscr{G}$ is a smooth affine $\mathfrak{o}$-group scheme then $\mathrm{H}^1(k^{\mathrm{un}}/k, \mathscr{G}(\mathfrak{o})) = 1$. Conclude that $\mathrm{H}^1(k, G) = 1$.
(4) Deduce that if G is any split semisimple k-group (maybe not simply connected), then $\mathrm{H}^1(k, G)$ is finite.

Exercise 5.17. Let G be a split connected reductive $\mathfrak{f}$-group. If $T \subset G$ is a split maximal $\mathfrak{f}$-torus, construct a natural bijection between maximal $\mathfrak{f}$-tori of G and conjugacy classes in the Weyl group $W(G, T)$ such that T_w is $\mathfrak{f}$-anisotropic if and only if w has no nonzero fixed points in $X^*(T)$.[94] Deduce that G admits an anisotropic maximal $\mathfrak{f}$-torus.[95]

Exercise 5.18. Let G be a split connected semisimple k-group. This exercise shows that G admits an anisotropic maximal k-torus.

(1) Show that if x is a special vertex of $\mathcal{B}(G)$, then $(\mathscr{G}_x)_{\mathfrak{f}}$ is reductive; such a point x is called *hyperspecial*.
(2) Use Exercise 5.17 and [Con14, Corollary B.2.6, Theorem B.3.2] to prove the claim.

Exercise 5.19. Let G be a split simply connected semisimple k-group, let $C \subset G$ be an étale central k-subgroup, and let $G' = G/C$, so there is a short exact sequence

$$1 \to C \to G \to G' \to 1$$

and hence a map $\delta \colon \mathrm{H}^1(k, G') \to \mathrm{H}^2(k, C)$. Using Exercises 4.61 and 5.18, show that δ is surjective.[96]

Remark 5.20. The conclusions of Exercises 5.16, 5.17, 5.18, and 5.19 all remain true if G is not assumed split, and the proofs are similar but more complicated; see [KP23, Theorem 10.5.1, Theorem 10.6.4, Theorem 10.6.8]. In fact, the map δ in Exercise 5.19 is bijective, but even in the split case proving injectivity requires the result of Exercise 5.16 when G is not necessarily split (since it involves the vanishing of cohomology for forms of G).

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[91] *Hint*: given a closed bounded subset Ω of $|\mathcal{B}(G)|$, use the axioms to show that there is a unique point in Ω with maximum distance to the boundary.

[92] For historical reasons, Steinberg's paper requires k to be perfect; an exposition of the proof Steinberg's theorem without this assumption can be found on my website.

[93] *Hint*: show that $\mathcal{I}$ is its own normalizer in $G(k)$, and use that any two chambers are $G(k)$-conjugate.

[94] *Hint*: if $gT_{\bar{\mathfrak{f}}}g^{-1} = T'_{\bar{\mathfrak{f}}}$, consider the element $F(g)g^{-1}$, where F is the Frobenius automorphism as in Exercise 3.38.

[95] *Hint*: use a Coxeter element; see [Bou68, Chapter V, §6.2, Def. 2, Thm. 1(i)].

[96] The étale assumption is not necessary, but without it one must move from Galois cohomology to flat cohomology, and the construction of δ is far from obvious.

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