AWS 2025: GEOMETRIZATIONS OF REPRESENTATIONS OF p-ADIC GROUPS

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1. Course Introduction

The goal of these lectures is to describe recent developments in building a geometric counterpart to the representation theory of p-adic groups. In this introduction, we give some brief comments on the starting point for our lecture course: the representation theory of finite groups of Lie type.

Finite groups of Lie type are groups of the form $\mathbb{G}(\mathbb{F}_q)$ where $\mathbb{G}$ is a connected reductive group and $\mathbb{F}_q$ is the finite field of $q = p^n$ elements—think $\mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{Sp}_{2n}(\mathbb{F}_q)$, $\mathrm{SO}(\mathbb{F}_q)$, as first examples. This class of groups has historical significance for many reasons, one of which is that it contains the most interesting of the three infinite classes of finite simple groups (the other two infinite classes are cyclic groups of prime order and alternating groups). In our present-day understanding, the representation theory of finite groups of Lie type is indivisible from geometry; this link is grounded in the study of *Deligne–Lusztig varieties.*

A ubiquitous link between geometry and representation theory comes from the basic notion from topology that if a space X has an action of a finite group G, then the cohomology groups $H^i(X)$ also have an action of G. In Deligne–Lusztig theory, X is a variety over $\overline{\mathbb{F}}_q$, $G = \mathbb{G}(\mathbb{F}_q)$, and the cohomology theory we use is ℓ-adic étale cohomology ($\ell \neq p$). In this way, from X we obtain a family of G-representations $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ with coefficients in the algebraically closed characteristic-zero field $\overline{\mathbb{Q}}_\ell$.

Definition 1.1. Let $\mathbb{G}$ be a connected reductive group over $\mathbb{F}_q$ and let $\sigma \colon \mathbb{G} \to \mathbb{G}$ be a Frobenius morphism. The *Deligne–Lusztig variety* associated to a maximal torus $\mathbb{T} \hookrightarrow \mathbb{G}$ and a Borel subgroup $\mathbb{B} \hookrightarrow \mathbb{G}_{\overline{\mathbb{F}}_q}$ containing $\mathbb{T}_{\overline{\mathbb{F}}_q}$, is the $\overline{\mathbb{F}}_q$-variety

$$X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}} := \{g \in \mathbb{G}(\overline{\mathbb{F}}_q) : g^{-1}\sigma(g) \in \mathbb{U}(\overline{\mathbb{F}}_q)\}.$$

This variety has an action of $\mathbb{G}(\mathbb{F}_q) \times \mathbb{T}(\mathbb{F}_q)$ given by $(\gamma, \tau) \cdot g = \gamma g \tau$. For a character $\theta \colon \mathbb{T}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$, We define its *Deligne–Lusztig induction* to be the virtual $\mathbb{G}(\mathbb{F}_q)$-representation

$$R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) := \sum_{i \geq 0} (-1)^i H^i_c(X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}, \overline{\mathbb{Q}}_\ell)_\theta,$$

where the subscript θ denotes the subspace in which $\mathbb{T}(\mathbb{F}_q)$ acts by multiplication by θ.

Let us describe the setting for $\mathbb{G} = \mathrm{SL}_2$. There are two possible $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$ that arise, corresponding to the split torus $\mathbb{F}_q^* \hookrightarrow \mathrm{SL}_2(\mathbb{F}_q)$ and the nonsplit torus $\ker(\mathrm{Nm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}) \hookrightarrow \mathrm{SL}_2(\mathbb{F}_q)$:

(1) (split: $\mathbb{T}(\mathbb{F}_q) \cong \mathbb{F}_q^\times$) $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$ is an affine fibration over the points $\mathbb{G}(\mathbb{F}_q)/\mathbb{U}(\mathbb{F}_q)$. Then $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) = \mathrm{Ind}_{\mathbb{B}(\mathbb{F}_q)}^{\mathrm{SL}_2(\mathbb{F}_q)}(\theta)$. These are called *principal series* representations.

(2) (nonsplit: $\mathbb{T}(\mathbb{F}_q) \cong \ker(\mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times)$). Then $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$ is an affine fibration over the affine curve $\mathbb{V}(x^q y - xy^q = 1)$. For $\theta \neq 1$, the representations arising in $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta)$ are called *cuspidal* representations.

The most basic example of the role of finite groups of Lie type in the representation theory of p-adic groups is the following 3-step recipe to construct a *depth-zero* supercuspidal representation of $\mathrm{SL}_2(F)$ for a non-archimedean local field F with ring of integers $\mathcal{O}_F$ and residue field $\mathbb{F}_q$.

(1) Let ρ be an irreducible cuspidal representation of $\mathrm{SL}_2(\mathbb{F}_q)$.
(These are understood by studying $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta)$.)
(2) Pull back ρ to a representation of $\mathrm{SL}_2(\mathcal{O}_F)$.
(3) Compactly induce: $\text{c-Ind}_{\mathrm{SL}_2(\mathcal{O}_F)}^{\mathrm{SL}_2(F)}(\rho)$ is irreducible supercuspidal.

Variations and elaborations of this 3-step recipe is at the core of developments in the representation theory of p-adic groups. In my lecture series, we will focus on variations of a geometric flavor.

2. Course Outline

Here is a tentative outline of lectures.

Lecture 1: Deligne–Lusztig theory. Foundational lecture on Deligne–Lusztig theory. References: [DL76]

Lecture 2: Positive-depth Deligne–Lusztig varieties. Parahoric Deligne–Lusztig varieties, relation to positive-depth representations of p-adic groups, L-packets. References: [Lus04, Sta09, CI21b, CI21a, CI23, CO24, Cha24].

Lecture 3: Character sheaves on parahoric subgroups. Character sheaves on connected reductive groups, character sheaves on parahoric subgroups, relation to Deligne–Lusztig theory. References: [Lus85, Lus90, BC24].

Lecture 4: Very regular elements. Characterization of (some) supercuspidal representations, relation to character sheaves on parahoric subgroups. [Lus20, Lus90, Hen92, CO23].

3. Projects

Masao Oi will be the project assistant. We will lead several projects, each of which will involve varying degrees of interplay between algebraic geometry and representations of p-adic groups, especially in wielding geometric constructions to investigate new representation-theoretic phenomena. Students in our project group should expect to arrive at the Arizona Winter School having read essential preliminary material, which will be communicated in due course.

References

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