GEOMETRIZATIONS OF REPRESENTATIONS OF p-ADIC GROUPS ARIZONA WINTER SCHOOL 2025

CHARLOTTE CHAN

These are notes to supplement and accompany my lectures at the Arizona Winter School on some emerging techniques geometrizing the representation theory of p-adic groups. They are not by any means meant to be complete, and in many cases, I would probably recommend studying the cited sources instead of this text! I have chosen to favor presenting material to fit a particular story arc for these lectures. As such, I strongly recommend the reader to only use these notes as a guide.

Contents

1. Lecture 1: Deligne–Lusztig theory

In the first lecture, I will tell you about the representation theory finite groups of Lie type. There are many excellent references on this topic: [DL76, Sri79, Car85, DM20, GM20]. In this lecture, and in these lecture notes, I will only discuss an overview of some aspects of Deligne–Lusztig theory that will be particularly relevant in later lectures of myself and the other lecturers.

1.1. **Introduction.** Let $\mathbb{G}$ be a connected reductive group over $\overline{\mathbb{F}}_q$ with a Frobenius root $\sigma\colon \mathbb{G} \to \mathbb{G}$. The finite group $\bar{G} := \mathbb{G}(\overline{\mathbb{F}}_q)^\sigma$ is called a *finite group of Lie type.* Connected reductive groups are classified by their associated root datum $(X^*, X_*, \Phi, \Phi^\vee)$, and this classification can be refined to classify finite groups of Lie type. For example, simple reductive $\mathbb{G}$ are classified by Dynkin diagrams, and the possible associated finite groups of Lie type $\bar{G}$ are classified by finite automorphisms of the above Dynkin diagrams:

split: A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, G_2,

non-split: ${}^2A_n$ $(n \geq 2)$, ${}^2D_n$ $(n \geq 4)$, ${}^3D_4$, ${}^2E_6$,

exceptional: ${}^2B_2(2^{2n+1})$ (Suzuki), ${}^2F_4(2^{2n+1})$ (Ree), ${}^2G_2(3^{2n+1})$ (Ree).

The split ones come from taking σ to be the Frobenius morphism, the non-split ones come from taking σ to be a Frobenius root induced by an automorphism of the Dynkin diagram, and the exceptional ones come from taking σ to be a Frobenius root induced by a non-length-preserving automorphism of the Dynkin diagram; in most of these cases, quotienting by the center yields a simple group. See [DM20, §4.3] for a much more detailed exposition. These form one of the three infinite families in the classification of finite simple groups.

Let us consider the case $\mathbb{G} = \mathrm{GL}_n$ for a moment. It seems not unreasonable to expect that the structure of the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$ would be reminiscent of the structure of the representation theory of $\mathrm{GL}_n(\mathbb{C})$. For $\mathrm{GL}_n(\mathbb{C})$, the finite-dimensional representations are in bijection with certain characters (one-dimensional representations) of the maximal torus $T(\mathbb{C})$ in $\mathrm{GL}_n(\mathbb{C})$. The first complication that appears when working with $\mathrm{GL}_n(\mathbb{F}_q)$ instead of $\mathrm{GL}_n(\mathbb{C})$ is that because $\mathbb{F}_q$ is not algebraically closed, $\mathrm{GL}_n(\mathbb{F}_q)$ has many maximal tori. They can be described explicitly: they are $\prod_i \mathbb{F}_{q^{n_i}}^\times$ where $\sum_i n_i = n$. Work of Green [Gre55] shows that still, it turns out that this expectation is almost correct: one has a map

$$\left\{\text{irreducible representations of } \mathrm{GL}_n(\mathbb{F}_q)\right\} \to \left\{\begin{matrix}\mathrm{GL}_n(\overline{\mathbb{F}}_q)\text{-conjugacy classes of} \\ \text{characters of maximal tori}\end{matrix}\right\}. \tag{1}$$

The "almost" in the preceding question reflects that this map is only injective on most of the source set.

In [DL76], Deligne and Lusztig establish (1) for an arbitrary finite group of Lie type.

Definition 1.1 (geometric conjugacy)**.** Let $\mathbb{T}$ and $\mathbb{T}'$ be two σ-stable maximal tori of $\mathbb{G}$, and let θ and θ' be characters of $\bar{T}$ and $\bar{T}'$. We say that $(\mathbb{T}, \theta)$ and $(\mathbb{T}', \theta')$ are *geometrically conjugate* if there is some n for which:

(1) $\mathbb{T}' = g\mathbb{T}g^{-1}$ for some $g \in \mathbb{G}(\overline{\mathbb{F}}_q)^{\sigma^n}$, and
(2) $\theta \circ \mathrm{Nm} = {}^g\theta' \circ \mathrm{Nm}$, where $\mathrm{Nm} = \mathrm{id} \cdot \sigma \cdot \sigma^2 \cdots \sigma^{n-1}\colon A(\overline{\mathbb{F}}_q)^{\sigma^n} \to A(\overline{\mathbb{F}}_q)^\sigma$ for any abelian algebraic group A with Frobenius root σ (in this case $A = \mathbb{T}$).

The parametrizing map for arbitrary finite groups $\bar{G}$ is then

$$\left\{\text{irreducible representations of } \bar{G}\right\} \to \left\{\begin{matrix}\text{geometric conjugacy classes of} \\ \text{characters of maximal tori}\end{matrix}\right\}. \tag{2}$$

Their construction proceeds via what we now call *Deligne–Lusztig induction*. In the next few subsections, I will tell you how this is defined, why it yields a map (2), and some of its important properties.

1.2. **Definitions.** Let $\mathbb{T}$ be a σ-stable maximal torus in $\mathbb{G}$ and choose a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$. Denote by $\mathbb{U}$ the unipotent radical of $\mathbb{B}$.

Definition 1.2. The *Deligne–Lusztig variety* associated to $\mathbb{T}, \mathbb{B}, \mathbb{G}$ is the $\overline{\mathbb{F}}_q$-scheme

$$X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}} := \{x \in \mathbb{G} : x^{-1}\sigma(x) \in \mathbb{U}\}.$$

It is easy to see that $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$ comes with two commuting group actions: $\bar{G}$ acts by left-multiplication (since every element of $\bar{G}$ is σ-fixed by definition) and $\bar{T}$ acts by right-multiplication (since $\mathbb{U}$ is normalized by $\mathbb{T}$ and hence $\bar{T}$). Explicitly, for $(g,t) \in \bar{G} \times \bar{T}$ and $x \in X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$,

$$(g,t) \cdot x = gxt.$$

Therefore, to obtain representations from $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$, we may consider its ℓ-adic étale cohomology groups $H_c^i(X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}, \overline{\mathbb{Q}}_\ell)$. We may sometimes write $H_c^* = \sum_{i \geq 0} (-1)^i H_c^i$.

Definition 1.3 (Deligne–Lusztig induction)**.** Let $\theta \colon \bar{T} \to \overline{\mathbb{Q}}_\ell^\times$ be a character. The virtual representation

$$R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) := \sum_{i \geq 0} (-1)^i H_c^i(X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}, \overline{\mathbb{Q}}_\ell)_\theta$$

is the *Deligne–Lusztig induction* of θ.

Example 1.4. It is useful to keep in mind the example $\mathbb{G} = \mathrm{GL}_2$. There are two conjugacy classes of maximal tori in $\mathrm{GL}_2(\mathbb{F}_q)$: $\bar{T}_e = \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ and $\bar{T}_w = \mathbb{F}_{q^2}^\times$. Allow us now to make very explicit choices, so as to make concrete the associated Deligne–Lusztig varieties and Deligne–Lusztig induction.

Let $\mathbb{T}$ denote the subgroup of diagonal matrices, let $\mathbb{B}$ denote the subgroup of upper triangular matrices, and let $\mathbb{U}$ denote the subgroup of upper triangular unipotent matrices. Then $\bar{T}_e$ and $\bar{T}_w$ correspond to the Frobenius roots σ_e and σ_w of $\mathbb{G}$:

$$\sigma_e \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^q & b^q \\ c^q & d^q \end{pmatrix},$$
$$\sigma_w \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^q & b^q \\ c^q & d^q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d^q & c^q \\ b^q & a^q \end{pmatrix}.$$

Note that

$$\bar{T}_e \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times, \qquad \bar{T}_w \cong \mathbb{F}_{q^2}^\times.$$

These are the split and nonsplit tori of $\mathbb{G}$. One sees that $\mathbb{B}, \mathbb{U}$ are σ_e-stable but not σ_w-stable.

Split torus. We can explicitly compute $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$ for σ_e. Since $\mathbb{U}$ is σ_e-stable, we know $\mathbb{U} \subset X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$. Since $\mathbb{U} \cong \mathbb{A}^1$, it is the case [DM20, Proposition 8.1.13] that the cohomology groups of $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$ and $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}/\mathbb{U}$ are equal after a degree shift by 2. We have

$$X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}/\mathbb{U} = \{x\mathbb{U} \in \mathbb{G}/\mathbb{U} : x^{-1}\sigma_e(x) \in \mathbb{U}\}.$$

Let $x \in \mathbb{G}$ be such that $x^{-1}\sigma(x) = u \in \mathbb{U}$. The Lang–Steinberg theorem [DM20, Theorem 4.2.9] says that for a connected algebraic group $\mathbb{H}$ and any Frobenius root σ_H on $\mathbb{H}$, the associated Lang–Steinberg map $h \mapsto h^{-1}\sigma_H(h)$ is surjective. Applying this to $\mathbb{U}$ and σ_e, we see that there exists a $u_0 \in \mathbb{U}$ such that $u_0^{-1}\sigma_e(u_0) = u$. Hence we have $x^{-1}\sigma_e(x) =$

$u_0^{-1}\sigma_e(u_0)$, which implies $(xu_0^{-1})^{-1}\sigma_e(xu_0^{-1}) = 1$. Of course $x\mathbb{U} = xu_0^{-1}\mathbb{U}$, and so we may now conclude

$$X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}/\mathbb{U} = \bar{G}\mathbb{U}/\mathbb{U} \cong \bar{G}/\bar{U}.$$

This is a 0-dimensional variety, and so

$$H_c^i(X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}/\mathbb{U}, \overline{\mathbb{Q}}_\ell) = \begin{cases} \overline{\mathbb{Q}}_\ell[\bar{G}/\bar{U}] & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, we now see

$$R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) = \overline{\mathbb{Q}}_\ell[\bar{G}/\bar{U}]_\theta = \{f \colon \bar{G}/\bar{U} \to \overline{\mathbb{Q}}_\ell \mid f(gt) = \theta(t)f(g) \text{ for all } g \in \bar{G}, t \in \bar{T}_e\}.$$

The space of functions on the right is immediately recognizable as an induced representation: setting $\hat{\theta} \colon \bar{B} \to \overline{\mathbb{Q}}_\ell^\times$ to be the pullback of θ along the natural surjection $\bar{B} \to \bar{T}$, we see that

$$R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) = \mathrm{Ind}_{\bar{B}}^{\bar{G}}(\hat{\theta}). \tag{3}$$

Nonsplit torus. Now let us compute $X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}$ for σ_w. We have

$$\begin{aligned} X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\overline{\mathbb{F}}_q) &= \left\{x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \sigma_w(x) \in x\mathbb{U}\right\} \\ &= \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a = d^q,\ c = b^q,\ ad - bc \in \mathbb{F}_q^\times\right\} \\ &\cong \left\{\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) : d^{q+1} - b^{q+1} \in \mathbb{F}_q^\times\right\}. \end{aligned}$$

The keen reader will notice that

$$\bar{G} = \left\{\left(\begin{smallmatrix} a & b^q \\ b & a^q \end{smallmatrix}\right) : a, b \in \mathbb{F}_{q^2},\ a^{q+1} - b^{q+1} \in \mathbb{F}_q^\times\right\},$$

which doesn't look like our standard presentation of $\mathrm{GL}_2(\mathbb{F}_q)$. The Lang–Steinberg theorem gives the existence of $h \in \mathrm{GL}_2(\mathbb{F}_{q^2})$ such that $h^{-1}\sigma_e(h) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and then conjugation by h gives an isomorphism

$$(\mathbb{G}, \sigma_w) \cong (\mathbb{G}, \sigma_e).$$

One can check explicitly that this induces an isomorphism $\bar{G} \cong \mathrm{GL}_2(\mathbb{F}_q)$ and

$$X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\overline{\mathbb{F}}_q) \cong \left\{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : xy^q - x^q y \in \mathbb{F}_q^\times\right\}.$$

This is a disjoint union of $q-1$ copies of the *Drinfeld curve.* One can compute (cite Bonnafe) that

$$H_c^i(X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}, \overline{\mathbb{Q}}_\ell)_\theta = \begin{cases} \theta_0 \circ \det & \text{if } \theta = \theta_0 \circ \det \text{ and } i = 2, \\ \mathrm{St}_{\bar{G}} \otimes \theta_0 \circ \det & \text{if } \theta = \theta_0 \circ \det \text{ and } i = 1, \\ \text{irreducible} & \text{if } \theta \neq \theta_0 \circ \det \text{ for any } \theta_0 \text{ and } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the irreducible representations $H_c^1(X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}, \overline{\mathbb{Q}}_\ell)_\theta$ when θ does not factor through the determinant map $\det \colon \mathrm{GL}_2(\mathbb{F}_q) \to \mathbb{F}_q^\times$, exactly comprise the irreducible representations of $\bar{G}$ which do not appear in any of the representations $\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\hat{\theta})$. That is to say:

1. (1) Every irreducible representation of $\mathrm{GL}_2(\mathbb{F}_q)$ appears in $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta)$ for some $(\mathbb{T}, \theta)$.
2. (2) For any character $\theta \colon \bar{T}_w \to \overline{\mathbb{Q}}_\ell^\times$ not factoring through det, we have the vanishing $\langle R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta), \mathrm{Ind}_{\bar{B}}^{\bar{G}}(\hat{\theta}_e)\rangle = 0$ for any character $\theta_e \colon \bar{T}_e \to \overline{\mathbb{Q}}_\ell^\times$. In other words, $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta)$ is *cuspidal.*

This allows us to see the first example of the role of parabolic induction in the representation theory of finite groups of Lie type. (In the p-adic setting, see Fintzen's notes and lectures for a discussion.) For $\mathrm{GL}_2(\mathbb{F}_q)$, it is simple: either an irreducible representation is cuspidal, or it appears in the parabolic induction from a maximal torus. (In general, one replaces the role of a maximal torus with a Levi.) $\diamond$

The argument presented in the σ_e case Example 1.4 generalizes without complications: If in fact there exists a σ-stable Borel subgroup $\mathbb{B} \subset \mathbb{G}$ containing $\mathbb{T}$, then

$$R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) = \mathrm{Ind}_{\bar{B}}^{\bar{G}}(\hat{\theta}),$$

where $\hat{\theta}$ is the pullback of θ along the natural map $\bar{B} \to \bar{T}$. The representation $\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\hat{\theta})$ is called the *parabolic induction* of θ to $\bar{G}$. Deligne–Lusztig induction should be thought of as a generalization of parabolic induction—even when a σ-stable $\mathbb{B}$ does not exist, Deligne–Lusztig induction allows us to associate to characters of $\bar{T}$ virtual representations of $\bar{G}$. It would therefore be prudent for the reader to keep in mind the special case of parabolic induction throughout the discussion of Deligne–Lusztig theory.

In general, parabolic induction is the induction from a representation $\hat{\rho}$ of a parabolic subgroup $\bar{P}$ of $\bar{G}$, where $\hat{\rho}$ is pulled back from a representation ρ of the Levi quotient of $\bar{P}$. We will revisit this discussion Section 1.5.

1.3. **The scalar product formula.** This section is centered around the discussion of the scalar product formula [DL76, Theorem 6.8]. In my lectures as well as others' lectures, it will be useful to consider $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta)$ for particularly nice θ. Let $\Phi(\mathbb{G}, \mathbb{T})$ denote the roots of $\mathbb{T}$ for $\mathbb{G}$ and consider the rational Weyl group $W_{\bar{G}}(\mathbb{T}) = \{g \in \bar{G} : g\mathbb{T}g^{-1} = \mathbb{T}\}/\bar{T}$.

Definition 1.5 (in general position, nonsingular)**.** Let θ be a character of $\bar{T}$.

(1) If θ has trivial stabilizer under the action of the Weyl group $W_{\bar{G}}(\mathbb{T})$, we say θ is *in general position*.
(2) Let $\mathbb{F}_q^n$ be the splitting field of $\mathbb{T}$ and set $\mathrm{Nm}\colon \mathbb{T}(\mathbb{F}_q)^{\sigma^n} \to \bar{T}$ be the group homomorphism given by $\mathrm{Nm}(t) = t\sigma(t)\cdots\sigma^{n-1}(t)$. If $\theta \circ \mathrm{Nm}|_{\alpha^\vee(\mathbb{F}_{q^n}^*)} \not\equiv 1$ for all roots $\alpha \in \Phi(\mathbb{G}, \mathbb{T})$, then we say that θ is *nonsingular*.

Note that if θ is in general position, then it is automatically nonsingular. If the center of $\mathbb{G}$ is connected, then all nonsingular θ are also in general position.

Theorem 1.6 (scalar product formula)**.** *Let $(\theta, \mathbb{T}, \mathbb{B}), (\theta', \mathbb{T}', \mathbb{B}')$ be arbitrary. Then*

$$\langle R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta), R_{\mathbb{T}',\mathbb{B}'}^{\mathbb{G}}(\theta)\rangle_{\bar{G}} = \sum_{w \in W_{\bar{G}}(\mathbb{T},\mathbb{T}')} \langle \theta, {}^w\theta'\rangle_{\bar{T}},$$

where $W_{\bar{G}}(\mathbb{T}, \mathbb{T}') = \{g \in \bar{G} : g\mathbb{T}g^{-1} = \mathbb{T}'\}/\bar{T}$ and ${}^w\theta'$ is the $\bar{T}$-character ${}^w\theta'(\dot{w}t'\dot{w}^{-1}) := \theta'(t')$ for any lift $\dot{w} \in \bar{G}$ of w.

First note that Theorem 2.9 also establishes that $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta)$ is independent of the choice of $\mathbb{B}$: indeed,

$$\langle R_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}(\theta), R_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}(\theta)\rangle = \langle R_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}(\theta), R_{\mathbb{T}_r,\mathbb{B}_r'}^{\mathbb{G}_r}(\theta)\rangle = \langle R_{\mathbb{T}_r,\mathbb{B}_r'}^{\mathbb{G}_r}(\theta), R_{\mathbb{T}_r,\mathbb{B}_r'}^{\mathbb{G}_r}(\theta)\rangle,$$

and therefore

$$\langle R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) - R_{\mathbb{T},\mathbb{B}'}^{\mathbb{G}}(\theta), R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) - R_{\mathbb{T},\mathbb{B}'}^{\mathbb{G}}(\theta)\rangle = 0,$$

which implies $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta) = R_{\mathbb{T},\mathbb{B}'}^{\mathbb{G}}(\theta)$. From now on, set

$$R_{\mathbb{T}}^{\mathbb{G}}(\theta) := R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta).$$

Remember that $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is only a virtual representation (a formal $\mathbb{Z}$-linear combination of irreducible representations) since it is defined as an alternating sum of cohomology groups. Therefore, the equation

$$\langle R_{\mathbb{T}}^{\mathbb{G}}(\theta), R_{\mathbb{T}}^{\mathbb{G}}(\theta)\rangle = 1$$

is equivalent to the statement that either $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is irreducible or it is the negative of an irreducible representation. Therefore Theorem 1.6 in particular gives us a criterion for the irreducibility of $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$, up to a sign:

Corollary 1.7. *$R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is (up to a sign) irreducible if and only if θ is in general position.*

A result with a similar flavor to Theorem 1.6 is the following, which at least for now we settle for just stating it:

Theorem 1.8. *If ρ is an irreducible representation which appears in $H_c^i(X_{\mathbb{T}}^{\mathbb{G}})_\theta$ and in $H_c^i(X_{\mathbb{T}'}^{\mathbb{G}})_{\theta'}$, then $(\mathbb{T}, \theta)$ and $(\mathbb{T}', \theta')$ are geometrically conjugate.*

1.4. **Deligne–Lusztig character formula.** In this subsection, we describe an explicit character formula for $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta)$. To do this, we will need the notion of the *Jordan decomposition*, which we now describe.

We say that an element $s \in \mathrm{GL}_n$ is *semisimple* if it is conjugate to a diagonal element, and we say that an element $u \in \mathrm{GL}_n$ is *unipotent* if it is conjugate to an upper-triangular unipotent element. This notion can be extended to arbitrary linear algebraic groups by choosing a closed embedding into some GL_n; it turns out that semisimplicity and unipotence do not depend on the choice of this embedding. An important property is that any element $g \in \mathrm{GL}_n$ can be written as the product of two commuting semisimple and unipotent elements: $g = su = us$. This is the Jordan decomposition; it is unique.

In the setting of $\bar{G}$, we can describe the above notions in very elementary terms, which for our purposes can be taken as definition:

Definition 1.9 (Jordan decomposition for $\bar{G}$)**.** An element $s \in \bar{G}$ is *semisimple* if it has prime-to-p order. An element $u \in \bar{G}$ is *unipotent* if it has p-power order. The Jordan decomposition $g = su = us \in \bar{G}$ is given by taken s and u to be the appropriate (Exercise: which?) powers of g. We write $\bar{G}_{\mathrm{unip}}$ for the set of unipotent elements in $\bar{G}$.

The Deligne–Lusztig character formula is based on the following general result:

Theorem 1.10 (Deligne–Lusztig fixed-point formula)**.** *Let X be a finite-type separated scheme and let g be any finite-order element of X. Let s,u be appropriate powers of g so that s has prime-to-p order and u had p-power order. Then*

$$\mathrm{Tr}(g; H_c^*(X, \overline{\mathbb{Q}}_\ell)) = \mathrm{Tr}(u; H_c^*(X^s, \overline{\mathbb{Q}}_\ell)).$$

Theorem 1.10 immediately tells us that the character of $R_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}(\theta)$ can be understood in terms of unipotent actions on the s-fixed points of Deligne–Lusztig varieties, as s varies over the semisimple elements of $\bar{G}$. This is structurally very nice: $(X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}})^s$ can be understood in terms of Deligne–Lusztig variety of the identity component of the centralizer of s. This is part of the content of the Deligne–Lusztig character formula (Theorem 1.11).

Theorem 1.11 (Deligne–Lusztig character formula)**.** *Let $\gamma = su$ be the Jordan decomposition for $\gamma \in \bar{G}$. Then*

$$\Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}(\gamma) = \frac{1}{|\bar{Z}_{\mathbb{G}}^0(s)|} \sum_{g \in \bar{G} \text{ s.t. } \mathbb{T}^g \subset \mathbb{Z}_{\mathbb{G}}^0(s)} \theta^g(s) \cdot \Theta_{R_{\mathbb{T}^g}^{\mathbb{Z}_{\mathbb{G}}^0(s)}(1)}(u).$$

Note in particular that the character value at any unipotent element is independent of the choice of θ! We may therefore make the following definition:

Definition 1.12 (Green function)**.** We call the following the *Green function*:

$$Q_{\mathbb{T},\mathbb{G}} := \Theta_{R_{\mathbb{T}}^{\mathbb{G}}}(\theta)|_{\bar{G}_{\mathrm{unip}}}.$$

1.5. **Construction of the map** (2)**.** A trick in [DL76] is the following proposition:

Proposition 1.13. *Let $s \in \bar{G}$ be a semisimple element. Then*

$$\frac{1}{\mathrm{St}_G(s)} \sum_{\mathbb{T} \subset Z_{\mathbb{G}}(s)} \sum_{\theta \in \mathrm{Irr}(\bar{T})} \theta(s)^{-1}(-1)^{r(\mathbb{G})-r(\mathbb{T})} \Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}(g) = \begin{cases} |Z_{\bar{G}}(s)| & \text{if } g \text{ is } \bar{G}\text{-conjugate to } s, \\ 0 & \text{otherwise} \end{cases}$$

where $r(\mathbb{G})$ and $r(\mathbb{T})$ denote the split ranks of $\mathbb{G}$ and $\mathbb{T}$, respectively.

We will not present its proof here (see [DL76, p. 141-142]). However, we will present some of its important consequences.

Theorem 1.14 (exhaustion)**.** *For any irreducible representation ρ of $\bar{G}$, there exists a $(\mathbb{T}, \theta)$ such that $\langle \rho, R_{\mathbb{T}}^{\mathbb{G}}(\theta) \rangle \neq 0$.*

Proof. Proposition 1.13 in the special case $s = e$ (the identity) implies that

$$|G| \cdot \delta_e = \frac{1}{\mathrm{St}_G(e)} \sum_{\mathbb{T} \subset \mathbb{G}} \sum_{\theta \in \mathrm{Irr}(\bar{T})} (-1)^{\sigma(\mathbb{G})-\sigma(\mathbb{T})} \Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}.$$

Since it is trivially the case that

$$\dim \rho = \Theta_\rho(e) = \langle \rho, |G| \cdot \delta_e \rangle,$$

we therefore have that

$$\dim \rho = \frac{1}{\mathrm{St}_G(s)} \sum_{\mathbb{T} \subset Z_{\mathbb{G}}(s)} \sum_{\theta \in \mathrm{Irr}(\bar{T})} (-1)^{\sigma(\mathbb{G})-\sigma(\mathbb{T})} \langle \rho, R_{\mathbb{T}}^{\mathbb{G}}(\theta) \rangle.$$

Obviously $\dim \rho$ is a positive number, so at least one of the summands must on the right-hand side of the above equation must be nonzero. ☐

We are finally able to construct the map (2). Let ρ be an irreducible representation of $\bar{G}$ and set

$$\Psi_\rho^\circ := \{(\mathbb{T}, \theta) : \langle \rho, R_{\mathbb{T}}^{\mathbb{G}}(\theta) \rangle \neq 0\}.$$

By Theorem 1.14, we know that Ψ_ρ° is nonempty. By Theorem 1.8, we know that Ψ_ρ° is contained in a geometric conjugacy class of $(\mathbb{T}, \theta)$; call this class Ψ_ρ. The map in (2) is given by the assignment

$$\rho \mapsto \Psi_\rho.$$

We make the following terminology:

(1) An irreducible representation ρ of $\bar{G}$ is *regular* if Ψ_ρ is in general position.
(2) An irreducible representation ρ of $\bar{G}$ is *nonsingular* if Ψ_ρ is nonsingular.
(3) An irreducible representation ρ of $\bar{G}$ is *unipotent* if Ψ_ρ contains the trivial character.

One can think of the irreducible representations of $\bar{G}$ on a spectrum, with unipotent representations on one extreme and regular representations on the other.

Remark 1.15. One might feel a bit disappointed: while the source and target of the map in (2) are completely elementary, the definition of this map involves very nontrivial geometry and cohomological results. It's natural to ask whether there is an elementary way to define this. In fact there is, under a largeness assumption on the size of $\mathbb{F}_q$! This is due to Lusztig [Lus20]. See Section 2.7 for a discussion of this topic. ⋄

In fact, the geometry of Deligne–Lusztig varieties allows us to further refine the above picture based on the notion of *parabolic induction.*

Proposition 1.16. *Let $\mathbb{P}$ be a σ-stable parabolic subgroup of $\mathbb{G}$ with Levi decomposition $\mathbb{P} = \mathbb{M}\mathbb{N}$. If $\mathbb{T}$ is a σ-stable maximal torus of $\mathbb{G}$ contained in $\mathbb{P}$, then for any character θ of $\bar{T}$, we have*

$$R_{\mathbb{T}}^{\mathbb{G}}(\theta) = \mathrm{Ind}_{\bar{P}}^{\bar{G}}(R_{\mathbb{T}}^{\mathbb{M}}(\theta)).$$

Definition 1.17 (cuspidal)**.** A representation π of $\bar{G}$ is called *cuspidal* if

$$\langle \pi, \mathrm{Ind}_{\bar{N}}^{\bar{G}}(1)\rangle = 0$$

for all σ-stable parabolic subgroups $\mathbb{P} = \mathbb{M}\mathbb{N}$ of $\mathbb{G}$. (Exercise: compare this to Definition 1.2.9 of Fintzen's notes.)

Theorem 1.18. *Let $\mathbb{T}$ be a σ-stable maximal torus in $\mathbb{G}$ such that $\mathbb{T}$ is not contained in any σ-stable proper parabolic subgroup of $\mathbb{G}$. If θ is non-singular, then $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is cuspidal.*

Proof. Choose any σ-stable parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$. Using Proposition 1.13 at $s = e$ and applying this to $\mathbb{M}$ implies that the character of $\mathrm{Ind}_{\bar{N}}^{\bar{G}}(1)$ can be written as a linear combination of $R_{\mathbb{T}'}^{\mathbb{G}}(\theta')$ as $\mathbb{T}'$ varies over all σ-stable maximal tori which are contained in $\mathbb{P}$. Since $\mathbb{T}$ is *not* contained in $\mathbb{P}$ by assumption, we have

$$\langle R_{\mathbb{T}}^{\mathbb{G}}(\theta), \mathrm{Ind}_{\bar{N}}^{\bar{G}}(1)\rangle = 0.$$

When θ is nonsingular, it turns out that (Theorem 1.19) $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is up to sign an actual representation of $\bar{G}$, so the conclusion follows. □

We record the result used in the proof of Theorem 1.18 (see [DL76, Proposition 7.4, Theorem 9.8]).

Theorem 1.19. *If θ is non-singular, then $(-1)^{r(\mathbb{G})-r(\mathbb{T})} R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is an honest $\bar{G}$-representation. In fact, $H_c^i(X_{\mathbb{T},\mathbb{B}}^{\mathbb{G}}, \overline{\mathbb{Q}}_\ell)_\theta$ is concentrated in a single degree.*

Remark 1.20. It is tempting to conclude after the displayed equation in the proof of Theorem 1.18 that then $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ is cuspidal for any θ, so long as $\mathbb{T}$ is not contained in any σ-stable proper parabolic. The first example of why this doesn't work happens for GL_2:

$$R_{\mathbb{T}}^{\mathrm{GL}_2}(1) = \begin{cases} \mathrm{St}_{\mathrm{GL}_2} + 1 & \text{if } \bar{T} \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times, \\ -\mathrm{St}_{\mathrm{GL}_2} + 1 & \text{if } \bar{T} \cong \mathbb{F}_{q^2}^\times. \end{cases}$$

So we see that $R_{\mathbb{T}}^{\mathrm{GL}_2}(1)$ is *not* cuspidal, but the inner product between these two Deligne–Lusztig inductions vanishes because of signs.

On the other hand, it is also not the case that Theorem 1.18 is sharp: there are cuspidal representations ρ of $\bar{G}$ for which Ψ_ρ is *not* nonsingular. The most famous example of this is the cuspidal unipotent representation of Sp_4 found by Srinivasan [Sri68], which is widely called θ_{10}, following Srinivasan's character table. ⋄

2. Lecture 2: Positive-depth Deligne–Lusztig varieties

2.1. **p-adic groups and parahoric subgroups.** Let F be a non-archimedean local field with residue field of size $q = p^n$. Let ϖ_F be a uniformizer for F and write $\mathcal{O}_F$ for the ring of integers of F. Denote by F^{ur} the maximal unramified extension of F. Let $\mathbf{G}$ be a connected reductive group over F and set $G = \mathbf{G}(F)$.

2.2. **Definitions.** Let $\mathbf{T}$ be an F-rational maximal torus in $\mathbf{G}$ which splits over an unramified extension of F. Let $\mathbf{B}$ be a F^{ur}-rational Borel subgroup of $\mathbf{G}_{F^{\mathrm{ur}}}$ containing $\mathbf{T}_{F^{\mathrm{ur}}}$; denote by $\mathbf{U}$ its unipotent radical.

Now choose an F-rational point $\mathbf{x}$ in the building of $\mathbf{G}$ which lies in the apartment of $\mathbf{T}$. As explained in [**?**], we have associated to $\mathbf{x}$ a compact open subgroup $G_{\mathbf{x},0} \subset G$ called a *parahoric subgroup*. Furthermore $G_{\mathbf{x},0}$ comes with a filtration of compact open subgroups $G_{\mathbf{x},r}$ (for $r \in \mathbb{R}_{\geq 0}$) known as the *Moy–Prasad filtration*; for $r < s$, we have $G_{\mathbf{x},s} \subseteq G_{\mathbf{x},r}$ and we define $G_{\mathbf{x},r+} := \bigcup_{s>r} G_{\mathbf{x},s}$. By [Yu15], there exists a nicely behaved group scheme $\mathcal{G}_{\mathbf{x},r}$ defined over $\mathcal{O}_F$ whose generic fiber is $\mathbf{G}$ and whose $\mathcal{O}_F$-points exactly realize $G_{\mathbf{x},r}$.

The algebraic group of interest for us will be a group scheme $\mathbb{G}_r$ defined over $\overline{\mathbb{F}}_q$, equipped with a Frobenius root σ, for which $\bar{G}_r := \mathbb{G}_r(\overline{\mathbb{F}}_q)^\sigma$ is the quotient $G_{\mathbf{x},0}/G_{\mathbf{x},r+}$. To define $\mathbb{G}_r$, we will use the $\mathcal{O}_F$-scheme $\mathcal{G}_{\mathbf{x},0}$ together with the "positive loops functor" L^+: for any $\mathcal{O}_F$-scheme X, define L^+X to be the $\mathbb{F}_q$-scheme defined by

$$L^+X(R) := X(\mathbb{W}(R))$$

for any $\overline{\mathbb{F}}_q$-algebra R. Here, $\mathbb{W}$ denotes the Witt ring associated to F if F has characteristic 0 ("mixed characteristic") and $\mathbb{W}(R) = R[t]$ if F has characteristic p ("equal characteristic"). It is well known that the Witt ring has good properties if R is perfect, but can behave poorly if R is not perfect; this same behavior is reflected in L^+. Because of this, it is commonplace to simply "pass to the perfection" when one is in the characteristic 0 setting; see [Zhu17, CI21b] for more details. (To be honest, very little is lost if the reader would prefer to think purely in the equal characteristic setting.)

Definition 2.1. Define $\mathbb{G}_r := (L^+\mathcal{G}_{\mathbf{x},0}/L^+\mathcal{G}_{\mathbf{x},r+})_{\overline{\mathbb{F}}_q}$. For any closed subgroup scheme $\mathbf{H} \subseteq \mathbf{G}_{F^{\mathrm{ur}}}$, one can associate a subgroup $\mathbb{H}_r \subseteq \mathbb{G}_r$ by considering the image in $\mathbb{G}_r$ of the schematic closure of $\mathbf{H}$ in $(\mathcal{G}_{\mathbf{x},0})_{\mathcal{O}_{F^{\mathrm{ur}}}}$ (see [CI21b, §2.6] for more details). In this way, associated to $\mathbf{T}_{F^{\mathrm{ur}}}, \mathbf{B}, \mathbf{U}$, we have subgroup schemes $\mathbb{T}_r, \mathbb{B}_r, \mathbb{U}_r$ of $\mathbb{G}_r$. The F-rationality of $\mathbf{x}$ endows $\mathbb{G}_r$ with a Frobenius root σ which stabilizes $\mathbb{T}_r$; we set $\bar{G}_r := \mathbb{G}_r(\overline{\mathbb{F}}_q)^\sigma$ and $\bar{T}_r := \mathbb{T}_r(\overline{\mathbb{F}}_q)^\sigma$.

Example 2.2. If F has equal characteristic, then in some cases

$$\mathbb{G}_r(R) = \mathbb{G}(R[t]/(t^{r+1}))$$

for a connected reductive group scheme $\mathbb{G}$. Some readers may recognize this construction—in this setting, $\mathbb{G}_r$ is called the *rth jet scheme* of $\mathbb{G}$. (This terminology and this construction works for any scheme, not just connected reductive algebraic groups.) Then $\mathbb{T}_r, \mathbb{B}_r, \mathbb{U}_r$ are also jet schemes of their counterparts in $\mathbb{G}$. In this context, positive-depth Deligne–Lusztig theory is a "jet" Deligne–Lusztig theory. ◇

Definition 2.3. The *positive-depth Deligne–Lusztig variety* associated to $\mathbf{T}, \mathbf{B}, \mathbf{G}, \mathbf{x}, r$ is the $\overline{\mathbb{F}}_q$-scheme

$$X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r} := \{x \in \mathbb{G}_r : x^{-1}\sigma(x) \in \mathbb{U}_r\}.$$

As in Section 1.2, it is easy to see that $X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}$ comes with two commuting group actions: $\bar{G}_r$ again acts by left-multiplication, and $\bar{T}_r$ again acts by right-multiplication, according to the same formula in *op. cit.*

Definition 2.4 (positive-depth Deligne–Lusztig induction)**.** Let $\theta\colon \bar{T}_r \to \overline{\mathbb{Q}}_\ell^\times$ be a character. The virtual representation

$$R_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}(\theta) := \sum_{i \geq 0} (-1)^i H_c^i(X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}, \overline{\mathbb{Q}}_\ell)_\theta$$

is the *Deligne–Lusztig induction* of θ.

Example 2.5 (GL_2, $r > 0$)**.** Suppose $\mathbf{G} = \mathrm{GL}_2$ and let $\mathbf{x}$ such that $G_{\mathbf{x},0} = \mathrm{GL}_2(\mathcal{O}_F)$. The Moy–Prasad filtration subgroups $G_{\mathbf{x},r}$ are exactly the congruence subgroups of $\mathrm{GL}_2(\mathcal{O}_F)$: the subgroup $G_{\mathbf{x},r}$ consists of all elements which are congruent to the identity modulo $\varpi_F^{\lceil r \rceil}$.

Furthermore assume that F has characteristic p so that $F = \mathbb{F}_q((t))$ and $F_2 = \mathbb{F}_{q^2}((t))$. Then $\mathbb{G}_r$ is the group scheme defined by

$$\mathbb{G}_r(R) = \mathrm{GL}_2(R[t]/t^{r+1})$$

for any $\overline{\mathbb{F}}_q$-algebra R. We can then give an explicit examples of $\mathbb{T}_r, \mathbb{B}_r, \mathbb{U}_r$, analogous to Example 1.4: let $\mathbb{T}_r$ be the subgroup of diagonal elements in $\mathbb{G}_r$, let $\mathbb{B}_r$ be the subgroup of upper-triangular elements in $\mathbb{G}_r$ so that $\mathbb{U}_r$ is the subgroup of upper-triangular elements with 1's along the diagonal. Then:

(1) The morphism $\sigma_e \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{smallmatrix}\right)$ corresponds to $\mathbf{T}$ being split.

(2) The morphism $\sigma_w \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \sigma(d) & \sigma(c) \\ \sigma(b) & \sigma(a) \end{smallmatrix}\right)$ corresponds to $\mathbf{T}$ being elliptic.

We leave it as an exercise to the reader to see that in the split case, one still has

$$X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}/\mathbb{U}_r \cong \bar{G}_r/\bar{U}_r$$

and in the elliptic case, one has

$$X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}(\overline{\mathbb{F}}_q) \cong \{\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) \in (\overline{\mathbb{F}}_q[t]/(t^{r+1}))^{\oplus 2} : d\sigma(d) - b\sigma(b) \in (\mathbb{F}_q[t]/(t^{r+1}))^\times\}.$$

One can then explicitly write out the defining equations of $X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}$. ◇

Example 2.6 ($\mathbf{T}$ split)**.** As in Section 1.2, when $\mathbf{T}$ is the split torus, then $X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}/\mathbb{U}_r \cong \bar{G}_r/\bar{U}_r$, and therefore

$$R_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}(\theta) = \mathrm{Ind}_{\bar{B}_r}^{\bar{G}_r}(\hat{\theta}),$$

where $\hat{\theta}$ is the pullback of θ along the natural map $\bar{B}_r \to \bar{T}_r$. ◇

At this point, it is reasonable to ask whether every statement presented in Section 1 also holds for the algebraic groups $\mathbb{G}_r$. The answer at present is "no". For example, it is not true that for any irreducible representation ρ of $\bar{G}_r$, there exists a pair $(\mathbb{T}_r, \theta)$ for which $\langle \rho, R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta) \rangle \neq 0$.

The representation theory of $\bar{G}_r$ is known to be very difficult. Only in some special cases have the irreducible representations of $\bar{G}_r$ have been classified (for example, Onn's work for GL_2 [Onn08]). In the notes here, we maintain the perspective of trying to understand what we can about $\bar{G}_r$ and its representation theory using positive-depth Deligne–Lusztig induction, especially keeping in mind the goal to link this picture with the representation theory of the p-adic group G.

For context, it is worth mentioning that these algebraic groups are rather interesting from a structural perspective: there is a natural surjection $\mathbb{G}_r \to \mathbb{G}_0$ to the connected reductive group $\mathbb{G}_0$, and the kernel of this map is a unipotent group. That is to say, $\mathbb{G}_r$ is an algebraic group that "combines" algebraic groups on opposite extremes of a spectrum—on the one hand, unipotent groups are p-groups, and on the other hand, we saw from Deligne–Lusztig theory that the representation theory of finite groups of Lie type is controlled by maximal tori, which are prime-to-p.

2.3. **Interlude: history, relation to representations of p-adic groups.** In the Corvallis proceedings, Lusztig [Lus79] proposed a p-adic version X_∞ of a Deligne–Lusztig variety and then analyzed it in the special case of the norm-1 elements of a division algebra. His analysis was of finite type: he realized X_∞ as an inverse limit over X_r for increasing r and then analyzed the cohomology of X_r.

Much later, in 2012, Boyarchenko formalized Lusztig's approach and studied this picture in the context of division algebras [Boy12], again by realizing X_∞ as (a disjoint union of) an inverse limit over certain varieties X_r. Surprisingly, the varieties X_1 associated to division algebras appear in a different context as well: in [BW16], Boyarchenko and Weinstein proved that X_1 is the special fiber of a particular open affinoid in the Lubin–Tate tower, and then use the cohomology of X_1 to prove that the cohomology of this special fiber realizes certain cases of the local Langlands correspondence.

Meanwhile, in 2004, Lusztig [Lus04] defined $X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}$ in the setting of jet schemes (see Example 2.2). This was generalized to the mixed characteristic setting using the Greenberg functor by Stasinski [Sta09]. However, neither Lusztig's nor Boyarchenko's X_r appears as a $X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}$.

In 2012, I had just started graduate school, and Boyarchenko suggested that I read his preprint [Boy12], especially to think about the two conjectures on division algebras presented there. I eventually resolved these two conjectures, first for dimension 4 and generic characters in equal characteristic [Cha16], then for arbitrary dimension and generic characters in equal characteristic [Cha18], and finally for arbitrary dimension and arbitrary characters in both equal and mixed characteristic [Cha20]. A key realization to handle the mixed characteristic setting was to realize that the scheme X_r could be reframed in the context of Iwahori subgroups—that is to say, that X_r was a $X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}$ in the setting that $\mathbb{G}_r$ arises from an Iwahori subgroup. This was the first time I realized the link between the p-adic picture and the developments of Lusztig and Stasinski. Of course, the $\mathbb{G}_r$ associated to Iwahori subgroups are *not* jet schemes, so the circumstances begged for a more general framework.

In [CI21b], Ivanov and I defined $X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}$ in the context presented in these notes: for $\mathbb{G}_r$ coming from parahoric subgroups associated to points in the building of $\mathbf{G}$ which lie in the apartments of unramified tori of $\mathbf{G}$. Our definition is a direct generalization of Lusztig's and Stasinski's definitions. Moreover, we also showed that the techniques introduced by Lusztig in [Lus04] essentially generalize flawlessly.

The question of how to define Lusztig's conjectural X_∞ is an important open problem. In the case of GL_n, this was resolved in [CI21a, CI23], for GSp and $\mathbf{T}$ Coxeter, this is work of Takamatsu [Tak23], and for general $\mathbf{G}$ and $\mathbf{T}$ Coxeter, this is work of Ivanov [Iva23a, Iva23b]. In all of these settings, X_∞ is the disjoint union of the inverse limit over $X_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}$; this is the geometric parallel to building a representation of G by taking the compact induction of the pullback of a representation of $\bar{G}_r$.

2.4. **The scalar product formula.** The focus of this section is on the following statement:

Conjecture 2.7 (Scalar Product Conjecture). *Fix $(\theta, \mathbb{T}_r, \mathbb{B}_r)$. For all $(\theta', \mathbb{T}_r', \mathbb{B}_r')$,*

$$\langle R_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}(\theta), R_{\mathbb{T}_r',\mathbb{B}_r'}^{\mathbb{G}_r}(\theta')\rangle = \sum_{w \in W_{\bar{G}_r}(\mathbb{T}_r,\mathbb{T}_r')} \langle \theta, {}^w\theta' \rangle.$$

When $r = 0$, this is Theorem 1.6. When $r > 0$, this conjecture is obviously false as stated. The simplest example is exactly the setting of Example 2.6: Suppose that $\mathbf{T}$ is the split torus and θ is a character of $\bar{T}_r$ which factors through a character of $\bar{T}_0$ in general position. Then although the right-hand side of Conjecture 2.7 is equal to 1 for any $r > 0$, the representation $\mathrm{Ind}_{\bar{B}_r}^{\bar{G}_r}(\hat{\theta})$ is only irreducible if $r = 0$ (exercise!).

There has been a lot of progress on Conjecture 2.7 in the last 20 years. We first introduce the following definition.

Definition 2.8 (weakly $(\mathbf{M}, \mathbf{G})$-generic)**.** Let $\mathbf{M}$ be a twisted Levi subgroup of $\mathbf{G}$ containing $\mathbf{T}$. We say that a character θ of $\bar{M}_r$ is *weakly* $(\mathbf{M}, \mathbf{G})$*-generic* if it satisfies the following condition: for every $\alpha \in \Phi(\mathbf{G}, \mathbf{T}) \backslash \Phi(\mathbf{M}, \mathbf{T})$, the restriction of the character $\theta \circ \mathrm{Nm}_{E/F}$ to $\alpha^\vee(E_r^\times)$ is nontrivial, where E is the splitting field of $\mathbf{T}$.

We briefly list the progress on Conjecture 2.7:

(1) True for θ weakly $(\mathbf{T}, \mathbf{G})$-generic and $(\mathbf{T}, \mathbf{B}, \mathbf{G})$ arbitrary [Lus04, Sta09, CI21b].
(2) True for θ arbitrary, $\mathbf{T}$ Coxeter, $\mathbf{B}$ chosen, $\mathbf{G}$ the norm-1 elements of a division algebra [Lus79].
(3) True for θ arbitrary, $\mathbf{T}$ Coxeter, $\mathbf{B}$ chosen, $\mathbf{G}$ a division algebra [Boy12, Cha20].
(4) True for θ arbitrary, $\mathbf{T}$ Coxeter, $\mathbf{B}$ chosen, $\mathbf{G}$ an inner form of GL_n [CI23].
(5) True for θ arbitrary, $\mathbf{T}$ Coxeter, $\mathbf{B}$ chosen, $\mathbf{G}$ arbitrary [DI20].
(6) True for θ arbitrary, $\mathbf{T}$ elliptic, $\mathbf{B}$ arbitrary, $\mathbf{G}$ arbitrary, p sufficiently large [Cha].

The approach in [Cha] is very different to the previous results on this. The technique uses Kaletha's results on *Howe factorizations*: if p is sufficiently large, then for any character θ of $T = \mathbf{T}(F)$, there exists a sequence of twisted Levi subgroups $\{\mathbf{G}^i\}$, and a sequence of $(\mathbf{G}^i, \mathbf{G}^{i+1})$-generic characters ϕ_i of $\mathbf{G}^i(F)$, and a depth-zero character ϕ_{-1} of $\mathbf{T}(F)$ such that $\theta = \prod_{i \geq -1} \phi_i|_T$ (see [Kal19] for more details). The sharp statement of the main theorem of [Cha] is:

Theorem 2.9. *Let $(\mathbb{T}_r, \mathbb{B}_r, \mathbb{G}_r)$ be arbitrary. If θ has a Howe factorization and is split-generic (i.e. the penultimate twisted Levi subgroup in the factorization does not properly contain a Levi subgroup of $\mathbf{G}$), then Conjecture 2.7 is true. In particular, the statement in (6) holds.*

Forthcoming work, using yet again a completely new approach to Conjecture 2.7 will establish an elementary formula for $\langle R_{\mathbb{T}_r, \mathbb{B}_r}^{\mathbb{G}_r}(\theta), R_{\mathbb{T}_r', \mathbb{B}_r'}^{\mathbb{G}_r}(\theta') \rangle$ and in particular show:

Theorem 2.10 (C. 2025, forthcoming)**.** *If $\mathbf{T}$ is elliptic, then Conjecture 2.7 holds.*

For the purposes of these notes, it will be enough for us to consider the setting of Theorem 2.9 for p sufficiently large, which assume for the rest of this section. First note that like in Section 1.3, Theorem 2.9 also establishes that $R_{\mathbb{T}_r, \mathbb{B}_r}^{\mathbb{G}_r}(\theta)$ is independent of the choice of $\mathbf{B}$. We may therefore set

$$R_{\mathbb{T}_r}^{\mathbb{G}}(\theta) := R_{\mathbb{T}_r, \mathbb{B}_r}^{\mathbb{G}_r}(\theta).$$

Definition 2.11. We say a character θ of $\bar{T}_r$ is *in general position* if it has trivial stabilizer in $W_{\bar{G}_r}(\mathbb{T}_r)$.

Note that this terminology is compatible with the $r = 0$ terminology by way of Kaletha's theory of Howe factorizations: if p is sufficiently large, then θ is in general position if and only if any Howe factorization $\theta = \prod_i \phi_i|_T$ of θ is such that ϕ_{-1} is in general position relative to $\mathbf{G}^0$.

We isolate the particular case of Theorem 2.9 which will be the linchpin of the discussion to come in the next subsections.

Corollary 2.12. *Assume* $\mathbf{T}$ *is elliptic. Then* $R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$ *is irreducible if and only if the* θ *is in general position.*

2.5. **Regular supercuspidal** L**-packets.** For this subsection, we assume that $\mathbf{T}$ is elliptic. Then then there is only a single F-rational point in the apartment of $\mathbf{T}$; this is our $\mathbf{x}$. We furthermore assume that p is sufficiently large so that every character of T has a Howe factorization. This exactly puts us in the context of Kaletha's regular supercuspidal representations.

The *regular depth-zero supercuspidal representations* of G are exactly the irreducible representations

$$\pi_{(\mathbf{T},\theta)}^{\mathrm{alg}} := \pi_{(\mathbf{T},\theta)}^{\mathrm{geom}} := \text{c-Ind}_{Z_G \cdot G_{\mathbf{x},0}}^{G}(R_{\mathbb{T}_0}^{\mathbb{G}_0}(\theta)),$$

where θ is in general position. Here, Z_G denotes the center of G and we view $R_{\mathbb{T}_0}^{\mathbb{G}_0}(\theta)$ as a representation of $G_{\mathbf{x},0}$ and then extend this representation to $Z_G G_{\mathbf{x},0}$ by demanding that Z_G act by $\theta|_{Z_G}$. Furthermore, it is a theorem of DeBacker and Reeder [DR09] and independently of Kazhdan and Varshavsky [KV06] that the assignment

$$(\mathbf{T}, \theta) \mapsto \pi_{(\mathbf{T},\theta)}^{\mathrm{alg}} = \pi_{(\mathbf{T},\theta)}^{\mathrm{geom}}$$

seems to be well behaved with respect to forming L-packets. This sets us up to ask the following two natural and urgently pressing questions:

(Q1) Is $\pi_{(\mathbf{T},\theta)}^{\mathrm{geom}} := \text{c-Ind}_{Z_G \cdot G_{\mathbf{x},0}}^{G}(R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta))$ irreducible when θ is in general position?
(Q2) Is $(\mathbf{T}, \theta) \mapsto \pi_{(\mathbf{T},\theta)}^{\mathrm{geom}}$ well behaved with respect to forming L-packets?

An affirmative answer to both questions, "100% of the time", is the content of my paper [CO24] with Oi and also of our forthcoming work. The goal of this subsection is to explain the context of these two questions and explain fundamental idea of our methodology.

Entirely independently of the considerations above, Kaletha [Kal19] proved that there is a particular class of supercuspidal representations which are parametrized by $(\mathbf{T}, \theta)$. The recipe proceeds as follows: a Howe factorization for $(\mathbf{T}, \theta)$ produces:

(1) a sequence $\vec{\mathbf{G}} := (\mathbf{G}^0, \mathbf{G}^1, \ldots, \mathbf{G}^d)$ of twisted Levi subgroups of $\mathbf{G}$
(2) a sequence $\vec{\phi} := (\phi_0, \phi_1, \ldots, \phi_d)$ of characters of increasing depth $r_0 < r_1 < \cdots < r_d$ where each ϕ_i is $(\mathbf{G}^i, \mathbf{G}^{i+1})$-generic
(3) a depth-zero character ϕ_{-1} of $\mathbf{T}$ in general position relative to $\mathbf{G}^0$

This exactly gives a collection of data from which Yu's construction [Yu01] produces a supercuspidal representation; denote this representation by $\pi_{(\mathbf{T},\theta)}^{\mathrm{alg}}$. The keen reader should be slightly suspicious: a Howe factorization for θ is not unique! ($\vec{\mathbf{G}}$ and $\vec{r}$ *are* uniquely determined by θ, but the sequence of characters $\vec{\phi}$ is not.) Indeed there is something to be checked. While Yu's construction associates to certain datum a supercuspidal representation, this assignment is not injective. Luckily, Hakim and Murnaghan [HM08] explicitly and cleanly describe the equivalence classes of datum producing the same supercuspidal, and Kaletha proves that any two Howe factorizations of $(\mathbf{T}, \theta)$ lie in the same such equivalence class. Hence the recipe described above is well defined.

The following is one of the main theorems of my forthcoming paper with Oi. (In a special case, this was established in our earlier work [CO24].)

Theorem 2.13. *Let* $(\mathbf{T}, \theta)$ *be in general position. If* $q \gg 0$*, then*

$$\pi^{\mathrm{geom}}_{(\mathbf{T},\theta)} \cong \pi^{\mathrm{alg}}_{(\mathbf{T},\theta\varepsilon^{\mathrm{ram}})},$$

where $\varepsilon^{\mathrm{ram}}$ *is a particular explicit sign character associated to* $(\mathbf{T}, \theta)$*. In particular, the answer to (Q1) is affirmative.*

One might be alarmed by the appearance of the sign character $\varepsilon^{\mathrm{ram}}$, but one should in fact have the opposite reaction. It was first discovered by DeBacker and Spice [DS18], in the so-called toral setting (i.e. when the Howe factorization of θ has $\vec{G} = (\mathbf{T}, \mathbf{G})$), that the assignment

$$(\mathbf{T}, \theta) \mapsto \pi^{\mathrm{alg}}_{(\mathbf{T},\theta)}$$

does *not* behave well with respect to forming L-packets in the sense that this map does *not* map stable conjugacy classes to stable character sums. However, they prove, under an additional assumption that F has characteristic zero with sufficiently large residual characteristic, that there exists an explicitly defined twisting character $\varepsilon^{\mathrm{ram}}$ for which the twisted assignment

$$(\mathbf{T}, \theta) \mapsto \pi^{\mathrm{alg}}_{(\mathbf{T},\theta\varepsilon^{\mathrm{ram}})}$$

does preserve stability [DS18, Theorem 5.10]. Kaletha extends the definition of $\varepsilon^{\mathrm{ram}}$ to relax the torality condition on θ. Kaletha's explicit local Langlands correspondence for regular supercuspidal representations is then defined using this twisted assignment.

From the perspective of the Langlands program, the content of Theorem 2.13 is that the geometric correspondence

$$(\mathbf{T}, \theta) \mapsto \pi^{\mathrm{geom}}_{(\mathbf{T},\theta)}$$

does not require an external twisting character: the geometry seems to innately know about the automorphic side of the local Langlands correspondence.

In summary, Theorem 2.13 additionally yields the following result:

Theorem 2.14. *If* $q \gg 0$*, the answer to (Q2) is also affirmative.*

In the next subsection, we will discuss the methodology of the proof of Theorem 2.13. The methodology is to come to a comparison of $\pi^{\mathrm{alg}}_{(\mathbf{T},\theta)}$ and $\pi^{\mathrm{geom}}_{(\mathbf{T},\theta)}$ by understanding the inducing datum, namely the representation on the parahoric subgroup before compactly inducing to the p-adic group.

Remark 2.15. At this point, we would like to highlight very interesting recent developments in understanding positive-depth Deligne–Lusztig induction. Work of Chen and Stasinski [CS17, CS23] gives an explicit description of $R^{\mathbb{G}_r}_{\mathbb{T}_r}(\theta)$ for $(\mathbf{T}, \mathbf{G})$-generic θ, up to a twist. Their approach is cohomological in nature. (Technically they don't exactly work in the context of the set-up in these notes, but this is a very minor point.) Even more recently, Nie [Nie24] generalized these ideas, combined them with the methodology of [Cha], to study $R^{\mathbb{G}_r}_{\mathbb{T}_r}(\theta)$ without the $(\mathbf{T}, \mathbf{G})$-genericity condition. ◇

Remark 2.16. An obvious question following Theorem 2.14 is: why? Or rather: is there a purely geometric explanation for the stability of the correspondence $(\mathbf{T}, \theta) \mapsto \pi^{\mathrm{geom}}_{(\mathbf{T},\theta)}$? For depth-zero θ, this was resolved by Bezrukavnikov and Varshavsky [BV21] by combining Lusztig's theory of character sheaves [Lus85] and a theorem of Yun [Yun14]. For positive-depth θ, this is ongoing work of Bezrukavnikov, Varshavsky, and myself. It is worth noting that these geometric methods establish endoscopic character identities in positive characteristic as well, which is a result not obtainable using the methods used by [DR09, DS18, FKS21] to study $(\mathbf{T}, \theta) \mapsto \pi^{\mathrm{alg}}_{(\mathbf{T},\theta\varepsilon^{\mathrm{ram}})}$. ◇

2.6. **Very regular elements.** The following definition was made in [CI21b], where we noticed that certain character values of $R_{\mathbb{T}_r,\mathbb{B}_r}^{\mathbb{G}_r}(\theta)$ were particularly simple. It is a generalization of terminology of Henniart [Hen92], who studied these elements and their remarkable properties in representation theory in the context of GL_n.

Definition 2.17. An element $\gamma \in G_{\mathbf{x},0}$ is called *very regular* if it satisfies the following conditions:

(1) γ is regular semisimple in $\mathbf{G}$
(2) the connected centralizer $\mathbf{T}_\gamma = Z_{\mathbf{G}}(\gamma)^\circ$ is an unramified torus in $\mathbf{G}$ such that $\mathbf{x}$ is an F-rational point in the apartment of $\mathbf{T}_\gamma$
(3) $\alpha(\gamma) \not\equiv 1 \pmod{\varpi_F}$ for any root α of $\mathbf{T}_\gamma$ in $\mathbf{G}$

The following theorem (Theorem 2.18) is remarkably useful. We will later use it to establish Theorem 2.13, the comparison result bridging the algebraic and geometric constructions of supercuspidals. The proof of Theorem 2.18 is remarkably simple. It is analytic in nature. While this actual theorem will only appear in our forthcoming paper, we have written down various incarnations of this theorem over the last several years.

Theorem 2.18 (C.–Oi, 2025)**.** *Assume $q \gg 0$ and let θ be a character of $\bar{T}_r$ in general position. There exists a unique irreducible representation π of $\bar{G}_r$ with character values*

$$\Theta_\pi(\gamma) = c \cdot \sum_{w \in W_{\bar{G}_r}(\mathbb{T}_r, \mathbb{T}_{\gamma,r})} \theta^w(\gamma), \qquad \text{for } \gamma \text{ very regular},$$

where $c \in \{\pm 1\}$ is a constant independent of the choice of γ.

Proof. Let π' be another irreducible representation of $\bar{G}_r$ satisfying the hypothesis of the theorem, and label the sign constant c'. Our task is to prove $\langle \pi, \pi' \rangle = 1$. To do this, it is enough to prove

$$\langle \pi, \pi' \rangle \neq 0. \tag{4}$$

Recall that $\langle -, - \rangle$ is defined as an average over the character values on $\bar{G}_r$. Denote by $\langle -, - \rangle_{\mathrm{vreg}}$ and $\langle -, - \rangle_{\mathrm{nvreg}}$ the partial averages over the character values on the very regular elements of $\bar{G}_r$ and the complement, respectively, so that $\langle -, - \rangle = \langle -, - \rangle_{\mathrm{vreg}} + \langle -, - \rangle_{\mathrm{nvreg}}$. By assumption, we know:

$$\langle \pi, \pi \rangle = 1 = \langle \pi', \pi' \rangle, \qquad \langle \pi, \pi \rangle_{\mathrm{vreg}} = cc' \langle \pi, \pi' \rangle_{\mathrm{vreg}} = \langle \pi', \pi' \rangle_{\mathrm{vreg}}.$$

Therefore

$$\langle \pi, \pi \rangle_{\mathrm{nvreg}} = \langle \pi', \pi' \rangle_{\mathrm{nvreg}}.$$

By the Cauchy–Schwarz inequality,

$$|\langle \pi, \pi' \rangle_{\mathrm{nvreg}}| \leq \langle \pi, \pi \rangle_{\mathrm{nvreg}}^{\frac{1}{2}} \cdot \langle \pi', \pi' \rangle_{\mathrm{nvreg}}^{\frac{1}{2}} = \langle \pi, \pi \rangle_{\mathrm{nvreg}}.$$

Now we make the following elementary observation: If $\langle \pi, \pi \rangle_{\mathrm{vreg}} > \frac{1}{2}$, then $\langle \pi, \pi \rangle_{\mathrm{nvreg}} < \frac{1}{2}$ (since these are averages over positive numbers). This then implies $|\langle \pi, \pi' \rangle_{\mathrm{vreg}}| > \frac{1}{2}$ and $|\langle \pi, \pi' \rangle_{\mathrm{nvreg}}| < \frac{1}{2}$ (from Cauchy–Schwarz), which then forces $\langle \pi, \pi' \rangle \neq 0$. Therefore, to prove (4), it is enough to prove

$$\langle \pi, \pi \rangle_{\mathrm{vreg}} > \frac{1}{2}.$$

We then compute the left-hand side:

$$\begin{aligned}\langle \pi, \pi \rangle_{\mathrm{vreg}} &= \frac{1}{|\bar{G}_r|} \sum_{\gamma \in (\bar{G}_r)_{\mathrm{vreg}}} \sum_{w, w' \in W_{\bar{G}_r}(\mathbb{T}_r, \mathbb{T}_{\gamma,r})} {}^w\theta(\gamma) \cdot \overline{{}^{w'}\theta(\gamma)} \\ &= \frac{1}{|\bar{G}_r|} \cdot \frac{|\bar{G}_r|}{|N_{\bar{G}_r}(\mathbb{T}_r)|} \sum_{t \in (\bar{T}_r)_{\mathrm{vreg}}} \sum_{w, w' \in W_{\bar{G}_r}(\mathbb{T}_r)} {}^w\theta(t) \cdot \overline{{}^{w'}\theta(t)} \\ &= \frac{1}{|N_{\bar{G}_r}(\mathbb{T}_r)|} \sum_{w, w' \in W_{\bar{G}_r}(\mathbb{T}_r)} \left(|\bar{T}_r| \langle {}^w\theta, {}^{w'}\theta \rangle - \sum_{t \in (\bar{T}_r)_{\mathrm{nvreg}}} {}^w\theta(t) \cdot \overline{{}^{w'}\theta(t)} \right) \\ &\geq \frac{1}{|N_{\bar{G}_r}(\mathbb{T}_r)|} \left(|\bar{T}_r| \cdot |W_{\bar{G}_r}(\mathbb{T}_r)| - |(\bar{T}_r)_{\mathrm{nvreg}}| \cdot |W_{\bar{G}_r}(\mathbb{T}_r)|^2 \right) \\ &= 1 - \frac{|(\bar{T}_r)_{\mathrm{nvreg}}|}{|\bar{T}_r|} \cdot |W_{\bar{G}_r}(\mathbb{T}_r)|.\end{aligned}$$

Since $|(\bar{T}_r)_{\mathrm{nvreg}}|$ is a polynomial in q of degree strictly less than the degree of $|\bar{T}_r|$, and therefore the ratio $\frac{|(\bar{T}_r)_{\mathrm{nvreg}}|}{|\bar{T}_r|}$ ends to 0 as $q \to \infty$. In particular, we see that $\langle \pi, \pi \rangle_{\mathrm{vreg}} > \frac{1}{2}$ for $q \gg 0$. ☐

The content of Theorem 2.18 is that an irreducible representation's character values on very regular elements can be used as a "litmus test." Let us demonstrate an application of this result by sketching a proof of Theorem 2.13.

Sketch of proof of Theorem 2.13. By [CI21b], we know that

$$\Theta_{R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)}(\gamma) = \sum_{w \in W_{\bar{G}_r}(\mathbb{T}_r, \mathbb{T}_{\gamma,r})} \theta^w(\gamma) \qquad \text{for } \gamma \in (\bar{G}_r)_{\mathrm{vreg}}.$$

By Corollary 2.12 we know that $R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$ is irreducible up to a sign (remember that $R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$ is an alternating sum of cohomology groups), so now $R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$ satisfies the hypothesis of Theorem 2.18. On the other hand, one can also compute the character formula for the irreducible representation ρ_θ on $\bar{G}_r$ arising from Yu's construction after applying Kaletha's Howe factorization to θ. One obtains

$$\Theta_{\rho_\theta}(\gamma) = c \sum_{w \in W_{\bar{G}_r}(\mathbb{T}_r, \mathbb{T}_{\gamma,r})} \varepsilon^{\mathrm{ram}}[(\mathbf{T}, \theta)]^w(\gamma) \cdot \theta^w(\gamma) \qquad \text{for } \gamma \in (\bar{G}_r)_{\mathrm{vreg}},$$

where $\varepsilon^{\mathrm{ram}}[(\mathbf{T}, \theta)]$ is a sign character which depends both on θ and on the torus $\mathbf{T} \subset \mathbf{G}$. Applying Theorem 2.18 now, we get

$$\rho_{\theta \cdot \varepsilon^{\mathrm{ram}}[(\mathbf{T},\theta)]} \cong c \cdot R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta).$$

(Note in particular that we see that whether $R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$ or $-R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$ is representable by an actual representation is exactly governed by the constant c.) By construction,

$$\pi_{(\mathbf{T},\theta)}^{\mathrm{alg}} = \text{c-Ind}_{Z_G G_{\mathbf{x},0}}^{G}(\rho_\theta), \qquad \pi_{(\mathbf{T},\theta)}^{\mathrm{geom}} = \text{c-Ind}_{Z_G G_{\mathbf{x},0}}^{G}(|R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)|),$$

so Theorem 2.13 now follows. ☐

At the time that Oi and I started thinking about (Q1) and (Q2) in Section 2.5, it wasn't at all clear how $\pi_{(\mathbf{T},\theta)}^{\mathrm{alg}}$ and $\pi_{(\mathbf{T},\theta)}^{\mathrm{geom}}$ were related. Comparing them, for example by equating their characters, seemed very daunting: character formulae are typically very sensitive to the construction of the representation, and as such, character formulae arising from tracing through algebraic methods and character formulae arising from cohomological origins are likely to have very different shapes. This is a prototypical situation where "litmus test" characterization theorems show their power: it allows one to discover phenomena "100% of the time" (all but finitely many cases) with only a limited amount of information. Giving a structural explanation for Theorem 2.13 is ongoing work of several in the field.

2.7. **Litmus tests for finite groups of Lie type and p-adic groups.** We end this section with an interlude on characterization theorems in a setting adjacent to the context of Sections 2.1 through 2.6.

To begin our discussion, we return to Remark 1.15. Deligne–Lusztig induction allows one to establish a map (2) associating each irreducible representation ρ to a geometric conjugacy class Ψ_ρ of characters of maximal tori. Is there an elementary way to define this map? In particular, is there a way to bypass cohomological considerations? The answer is yes when $q \gg 0$; this was proved by Lusztig in 1977 [Lus20] (see also [CO23, §4.2.2]).

Theorem 2.19 (Lusztig, 1977)**.** *If $q \gg 0$, then $\rho \mapsto \Psi_\rho$ is uniquely determined by $\Theta_\rho|_{\bar{G}_{\mathrm{rss}}}$.*

We isolate a particular setting of interest. Recall that an irreducible representation ρ of $\bar{G}$ is called unipotent if $\langle \rho, R_{\mathbb{T}}^{\mathbb{G}}(1)\rangle \neq 0$ for some maximal torus $\mathbb{T} \subset \mathbb{G}$. In practice this is not such an easy condition to check. Lusztig's methods give a simple criterion:

Theorem 2.20. *If $q \gg 0$, then an irreducible representation ρ is unipotent if and only if $\Theta_\rho|_{\bar{T}_{\mathrm{rss}}}$ is constant for every maximal torus $\mathbb{T} \subset \mathbb{G}$.*

For many years, the question of whether supercuspidal representations can be recognized from their character values on a special locus has been circulated. The first instance of such a theorem is due to Henniart in the 1990s [Hen92, Hen93] for certain tori GL_n. Henniart coined the terminology *très régulier* and established characterization theorem for certain supercuspidal representations which he then used to compare two different constructions of these representations (algebraic vs. trace formula). Whether such a theorem could exist for general $\mathbf{G}$ was posed by Adler and Spice [AS09] in the introduction of their paper on supercuspidal character formulae. One compelling motivation to want an answer to this question is the apparently intricate nature of complete character formulae of supercuspidal representations. Another compelling motivation to want an answer to this question is that it would give a p-adic analogue of Harish-Chandra's characterization of discrete series representations of real groups [HC65].

In [CO23], we answer this question. The first-approximation statement of our main theorem is:

Theorem 2.21. *If there are enough very regular elements, then the character values of a supercuspidal representation π nearly recovers its Yu-datum.*

Here, let me present specializations of this theorem on the two extreme ends of the spectrum: for regular supercuspidal representations (supercuspidals whose Yu-datum has a regular depth-zero representation) and for unipotent supercuspidal representations (depth-zero supercuspidals whose Yu-datum has a unipotent depth-zero representation).

First, a characterizing theorem for regular supercuspidal representations:

Theorem 2.22 (C.–Oi, 2023)**.** *Let $\mathbf{T} \subset \mathbf{G}$ be a tamely ramified elliptic maximal torus of $\mathbf{G}$. If T has enough very regular elements, then for any character θ in general position, the associated supercuspidal $\pi_{(\mathbf{T},\theta)}$ is uniquely determined by its character values on very regular elements.*

Next, the p-adic analogue of Lusztig's characterization of unipotent representations for finite groups of Lie type:

Theorem 2.23 (C.-Oi, 2023)**.** *If $q \gg 0$, then an irreducible supercuspidal representation π is unipotent if and only if*

(i) $\Theta_\rho|_{T_{\mathrm{vreg}}}$ *is constant for every maximally unramified elliptic maximal torus* $\mathbf{T} \subset \mathbf{G}$
(ii) *there exists a maximally unramified elliptic maximal torus* $\mathbf{T} \subset \mathbf{G}$ *such that* $\Theta_\pi|_{T_{\mathrm{vreg}}} \neq 0$.

Theorem 2.23 answers a question of DeBacker.

3. Lecture 3: Character sheaves

Up to this point in these notes, the relationship between representation theory and algebraic geometry came from the following basic principle:

(1) $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ is a $\overline{\mathbb{Q}}_\ell$-vector space.
(2) Functoriality of H_c^i implies that endomorphisms of X yield endomorphisms of $H_c^i(X)$.
(3) Therefore if X has an action by some group, then $H_c^i(X)$ is a representation of that group.

We in particular applied the above basic construction in the following setting: If $X = L_q^{-1}(Y)$ for a subvariety $Y \subset H$ of an algebraic group H with Frobenius root σ and associated Lang–Steinberg map L_q, then X has a left-multiplication action by $\bar{H}$ and a right-multiplication action by a subgroup of $\bar{H}$ which normalizes Y. In other words, we used interesting varieties to obtain interesting representations.

For the rest of these notes, we discuss a different incarnation of the relationship between representation theory and algebraic geometry: we will use interesting sheaves to obtain characters of interesting representations. The basic underlying principle here is the *sheaves-to-functions correspondence.*

Definition 3.1 (sheaf-to-function)**.** Let X be a scheme of finite type over $\mathbb{F}_q$ with associated Frobenius σ and let $\mathcal{F}$ be a complex of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on X with respect to the étale topology. For each closed point $x \in X$, the geometric Frobenius Frob_x at x acts on the geometric stalk $\mathcal{F}_{\bar{x}}$, a complex of $\overline{\mathbb{Q}}_\ell$-vector spaces. The function on $X(\mathbb{F}_q)$ corresponding to $\mathcal{F}$ is the function

$$\Theta_{\mathcal{F},\sigma} \colon X(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell, \qquad x \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}(\mathrm{Frob}_x, H_c^i(\mathcal{F}_{\bar{x}})).$$

This correspondence behaves well with respect to various sheaf operations: pullback corresponds to pullback, pushforward corresponds to sum. These lead to other nice properties: base-change corresponds to change-of-variables, projection formula corresponds to factoring. It goes the other way as well: if you have a nice respectable operation on functions, chances are you have a nice respectable operation on sheaves. For example, Fourier transform associates a group homomorphism $\psi \colon \mathbb{F}_p \to \mathbb{C}^\times$ to an indicator function $\delta_a \colon \mathbb{F}_p \to \mathbb{C}$. In sheafy land, the notion of Fourier transform exists as well, and associates a multiplicative local system (Artin–Schrier sheaf in this setting) on $\mathbb{G}_a$ to a skyscraper sheaf.

sheaf	function
multiplicative local system	multiplicative character
pullback	pullback
pushforward	average (sum)
base-change theorem	change-of-variables
projection formula	factoring out
Fourier transform	Fourier transform
convolution	convolution

Of course, given any representation, its character is a *conjugation-invariant* function on the group. So, we will be particularly interested in the setting that: (a) X is a algebraic group, and (b) $\mathcal{F}$ has some additional symmetry yielding a conjugation-invariant $\Theta_{\mathcal{F},q}$. The notion controlling (b) is *conjugation equivariance.*

3.1. **Equivariant derived categories.** Let X be a finite-type $\mathbb{F}_q$-scheme endowed with an action of an algebraic group H defined over $\mathbb{F}_q$. We denote by $\mathrm{Sh}(X)$ the category of constructible sheaves on X and we denote by $\mathcal{D}(X)$ the bounded derived category of $\mathrm{Sh}(X)$. The category $\mathcal{D}(X)$ has a triangulated structure and we denote by $\mathcal{P}(X)$ the heart of $\mathcal{D}(X)$ with respect to the perverse t-structure; objects in $\mathcal{P}(X)$ are called *perverse sheaves*. Giving an introduction to perverse sheaves is beyond the scope of these lectures, so we defer the reader to the literature (for example, consider the expositions [HTT08, Gor, Wil, Ach21]).

The H-action on X can be used to define H-equivariant versions $\mathrm{Sh}_H(X)$, $\mathcal{D}_H(X)$, and $\mathcal{P}_H(X)$ of $\mathrm{Sh}(X)$, $\mathcal{D}(X)$, and $\mathcal{P}(X)$. Let $a \colon H \times X \to X$ denote the action map.

Definition 3.2. A sheaf $K \in \mathrm{Sh}(X)$ is called *H-equivariant* if:

(i) (compatible with action) there is an isomorphism $\varphi \colon a^*\mathcal{F} \to p_2^*\mathcal{F}$, where $a \colon H \times X \to X$ is the action map and $p_2 \colon H \times X \to X$ is the projection to the 2nd coordinate.
(ii) (associativity) the isomorphism φ satisfies $b^*\varphi \circ \mathrm{pr}_{23}^*\varphi = m^*\varphi$, where $m, b, \mathrm{pr}_{23} \colon H \times H \times X \to H \times X$ are the maps $m(h_1, h_2, x) = (h_1h_2, x)$, $b(h_1, h_2, x) = (h_1, a(h_2, x))$, and $\mathrm{pr}_{23}(h_1, h_2, x) = (h_2, x)$.

The definition of $\mathcal{P}_H(X)$ is completely analogous to the definition of $\mathrm{Sh}_H(X)$. One would like to construct a triangulated category $\mathcal{D}_H(X)$ whose perverse heart is $\mathcal{P}_H(X)$. This turns out to be not so trivial: the naive guess of taking $\mathcal{D}_H(X)$ to be the derived category of the category of H-equivariant sheaves on X does *not* work. Bernstein and Lunts [BL94] resolve this: the correct definition comes from replacing X by an appropriate $\tilde{X}$ on which H acts freely and using the naive definition in this free setting. For more details, see for example [Ach21, Chapter 6].

For these notes, it will only be important that such a construction exists for us to use. We will want, for example: for any H-equivariant morphism $X \to Y$, the functors f_*, $f_!$, f^*, $f^!$ lift canonically to functors between $\mathcal{D}_H(X)$ and $\mathcal{D}_H(Y)$, and the usual properties of these functors (e.g. adjunction) are preserved. Of particular importance to us will be the special case that X is an algebraic group $\mathbb{G}$ and considered with the conjugation action of $\mathbb{G}$.

Warning 3.3. At present, in these notes, I have given no attention at all to shifts. So, all statements (e.g. about perversity) should be taken *up to a shift*.

3.2. **Character sheaves on connected reductive groups.** We give a brief introduction to some aspects of Lusztig's theory of character sheaves on connected reductive groups.

Let $\mathbb{G}$ be a connected reductive group over $\overline{\mathbb{F}}_q$. Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$ and let $\mathbb{B}$ be a Borel subgroup in $\mathbb{G}$ containing $\mathbb{T}$. The *Grothendieck–Springer resolution* of $\mathbb{G}$ is

$$\widetilde{\mathbb{G}} := \{(g, h\mathbb{B}) \in \mathbb{G} \times \mathbb{G}/\mathbb{B} : h^{-1}gh \in \mathbb{B}\}.$$

Of tremendous importance are the two maps

$$\begin{array}{ccc} & \widetilde{\mathbb{G}} & \\ {}^{f}\swarrow & & \searrow^{\pi} \\ \mathbb{T} & & \mathbb{G} \end{array} \qquad \begin{aligned} f(g, h\mathbb{B}) &= \mathrm{pr}_{\mathbb{T}}(h^{-1}gh), \\ \pi(g, h\mathbb{B}) &= g. \end{aligned}$$

Definition 3.4. $\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}} := \pi_! f^*$ gives a functor $\mathcal{D}_{\mathbb{T}}(\mathbb{T}) \to \mathcal{D}_{\mathbb{G}}(\mathbb{G})$ called *parabolic induction*.

The following example justifies this terminology.

Example 3.5. Suppose that σ is a Frobenius root on $\mathbb{G}$ which stabilizes both $\mathbb{T}$ and $\mathbb{B}$. There is a bijection between characters $\theta\colon \bar{T} \to \overline{\mathbb{Q}}_\ell^\times$ and σ-equivariant multiplicative local systems on $\mathbb{T}$: for each θ, there exists a unique σ-equivariant multiplicative local system on $\mathbb{T}$ such that $\Theta_{\mathcal{L},\sigma} = \theta$. Then for any $g \in \bar{G}$:

$$\Theta_{\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}),\sigma}(g) = \sum_{\substack{h\mathbb{B} \in \mathbb{G}/\mathbb{B} \text{ s.t.} \\ \sigma(h\mathbb{B})=h\mathbb{B}, \text{ and} \\ h^{-1}gh\in\mathbb{B}}} \theta(\mathrm{pr}_{\mathbb{T}}(h^{-1}gh)) = \Theta_{\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\hat{\theta})}(g),$$

where $\hat{\theta} = \theta \circ \mathrm{pr}_{\mathbb{T}}\colon \bar{B} \to \overline{\mathbb{Q}}_\ell^\times$ as in Example 1.4. Here, the first equality holds using the sheaves-to-functions dictionary, and the second equality holds via a basic character calculation. ◇

An important class of morphisms in the theory of perverse sheaves is the class of *semismall morphisms*, i.e. morphisms $\pi\colon X \to Y$ satisfying

$$\mathrm{codim}_Y\{y \in Y : \dim \pi^{-1}(y) \geq d\} > 2d \qquad \text{for } d \in \mathbb{Z}_{\geq 1}.$$

This is because of the following theorem (see [Ach21, Theorem 3.8.4]):

Theorem 3.6. *Assume X is smooth and connected and let $f\colon X \to Y$ be a proper morphism. If f is semismall, then for any local system $\mathcal{L}$ on X, the pushforward $f_*\mathcal{L}[\dim X]$ is perverse.*

It is clear that π is proper ($\mathbb{G}/\mathbb{B}$ is). It turns out that π is also semismall; this is a key starting proposition for the theory of character sheaves. The following dimension calculation (see [Lau89, Lemma 1.2.1]) (nearly) shows (interesting exercise!) that π is semismall:

Proposition 3.7. *The fiber product $\widetilde{\mathbb{G}} \times_{\mathbb{G}} \widetilde{\mathbb{G}}$ with respect to π has dimension* $\dim \mathbb{G}$.

The upshot of this discussion is:

Theorem 3.8. *For any local system $\mathcal{L}$ on $\mathbb{T}$, the parabolic induction $\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L})$ is perverse.*

In [Lus85], Lusztig defines a class of $\mathbb{G}$-equivariant perverse sheaves on $\mathbb{G}$ and calls them *character sheaves*. We won't give the definition here, but we give a class of examples:

Definition 3.9 (some character sheaves)**.** For any local system $\mathcal{L}$ on $\mathbb{T}$, the parabolic induction $\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L})$ is a direct sum of irreducible character sheaves.

Remark 3.10. As we know from classical representation theory, parabolic induction refers to induction of a representation on any parabolic subgroup which factors through the Levi quotient. The Borel/torus definition of pInd in Definition 3.4 can be generalized to a parabolic/Levi definition in this geometric context. In this more general setting, Theorem 3.8 still holds (see [MS89, §9.3]):

> *Theorem* 3.11. *If $\mathcal{K}$ is a character sheaf on the Levi quotient $\mathbb{M}$ of a parabolic $\mathbb{P}$ of $\mathbb{G}$, then $\mathrm{pInd}_{\mathbb{M}}^{\mathbb{G}}(\mathcal{K})$ is a character sheaf on $\mathbb{G}$.*

We will return to this in remark when we discuss character sheaves on $\mathbb{G}_r$. ◇

An important special case to keep in mind is that the irreducible character sheaves on $\mathbb{T}$ are exactly the multiplicative local systems on $\mathbb{T}$. In particular, Theorem 3.8 proves:

Theorem 3.12. *If $\mathcal{L}$ is a character sheaf on $\mathbb{T}$, then $\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L})$ is a character sheaf on $\mathbb{G}$.*

Furthermore, we have the following:

Theorem 3.13. *If $\mathcal{L}$ has trivial stabilizer in $W_{\mathbb{G}(\overline{\mathbb{F}}_q)}(\mathbb{T})$, then $\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L})$ is simple.*

At this point, we have a construction (sheafy parabolic induction) that inputs a character sheaf on $\mathbb{T}$ and outputs a character sheaf on $\mathbb{G}$. Recall that Deligne–Lusztig induction inputs a character on a maximal torus of $\bar{G}$ and outputs a (virtual) representation of $\bar{G}$. Both of these settings have a notion of trace: on the sheafy side, we have the sheaves-to-functions correspondence, and on the representation side, we have ordinary trace in the sense of linear algebra. We showed already (Example 3.5) that in the split setting (i.e. when Deligne–Lusztig induction corresponds to ordinary parabolic induction), these two constructions coincide:

$$\begin{array}{ccccc} & & \theta & & \\ & \swarrow & & \searrow & \\ \mathrm{pInd}(\mathcal{L}_\theta) & & & & R_{\mathbb{T}}^{\mathbb{G}}(\theta) \\ & \searrow^{\text{sh-to-fn}} & & \swarrow^{\text{trace}} & \\ & & \Theta_{\mathrm{pInd}(\mathcal{L}_\theta),\sigma} = \Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)} & & \end{array} \tag{5}$$

where $\mathcal{L}_\theta$ is the multiplicative local system on $\mathbb{T}$ corresponding to θ.

In [Lus90], Lusztig proved that for $q \gg 0$, this diagram in fact makes sense for Deligne–Lusztig induction in general:

Theorem 3.14. *Let σ be a Frobenius root on $\mathbb{G}$ and let $\mathbb{T}$ be a σ-stable maximal torus. For a character $\theta \colon \bar{T} \to \overline{\mathbb{Q}}_\ell^\times$, let $\mathcal{L}_\theta$ denote its corresponding multiplicative local system on $\mathbb{T}$. Then*

$$\Theta_{\mathrm{pInd}(\mathcal{L}_\theta),\sigma} = (-1)^{\dim \mathbb{T}} \cdot \Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}.$$

Remark 3.15. In fact, Lusztig proved much more in [Lus90]: he proved that even for the twisted version of general parabolic induction, the above compatibility holds. For an exposition of this work in the special case of a torus described above, one can read [Lau89] for the subsequent simplifications of the arguments of [Lus90]. ◇

3.3. **Character sheaves on parahoric subgroups.** Lusztig's character sheaves on reductive groups is one of the most influential subjects in representation theory. In [Lus06], Lusztig conjectured the existence of:

(1) a theory of character sheaves on unipotent groups,
(2) at least a "generic" theory of character sheaves on jet schemes of reductive groups.

The foundations of (1) were resolved by work of Boyarchenko and Boyarchenko–Drinfeld [BD06, Boy10, Boy11, Boy13, BD14]. Lusztig observed in [Lus06] that one cannot hope to have a complete theory of character sheaves on an arbitrary connected affine algebraic group in the sense that it may be the case that the functions associated to the conjugation-equivariant perverse sheaves only span a subspace of the space of class functions. However, Lusztig observed that perhaps in the setting of (2), some of the constructions in the setting of reductive groups have natural analogues. The purpose of the next few subsections is to describe recent work [BC24] of mine with R. Bezrukavnikov resolving (2). Let us begin by stating Lusztig's conjecture. We work with the algebraic groups $\mathbb{G}_r$ defined in Section 2.

Define

$$\widetilde{\mathbb{G}}_r := \{(g, h\mathbb{B}_r) \in \mathbb{G}_r \times \mathbb{G}_r/\mathbb{B}_r : h^{-1}gh \in \mathbb{B}_r\}.$$

Analogously to the $r = 0$ setting discussed in Section 3.2, one has two maps

$$\begin{array}{ccc} & \widetilde{\mathbb{G}}_r & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle \pi} \\ \mathbb{T}_r & & \mathbb{G}_r \end{array} \qquad\qquad \begin{aligned} f(g, h\mathbb{B}_r) &= \mathrm{pr}_{\mathbb{T}_r}(h^{-1}gh), \\ \pi(g, h\mathbb{B}_r) &= g. \end{aligned}$$

As before, we define:

Definition 3.16. $\mathrm{pInd}_{\mathbb{T}_r}^{\mathbb{G}_r} := \pi_! f^*$ is a functor $\mathcal{D}_{\mathbb{T}_r}(\mathbb{T}_r) \to \mathcal{D}_{\mathbb{G}_r}(\mathbb{G}_r)$ called *parabolic induction.*

Definition 3.17. We say that a multiplicative local system $\mathcal{L}_\psi$ on $\mathfrak{t} := \ker(\mathbb{T}_r \to \mathbb{T}_{r-1})$ is *weakly* $(\mathbf{T}, \mathbf{G})$*-generic* if it satisfies the following condition: for every $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, the restriction $\mathcal{L}_\psi|_{\ker(\mathbb{T}_r^\alpha \to \mathbb{T}_{r-1}^\alpha)}$ is a nontrivial local system, where $\mathbf{T}^\alpha = \alpha^\vee(\mathbb{G}_m)$. We say a weakly $(\mathbf{T}, \mathbf{G})$-generic $\mathcal{L}_\psi$ is $(\mathbf{T}, \mathbf{G})$*-generic* if it has trivial stabilizer in the absolute Weyl group.

Since $\mathbb{T}_r$ is an abelian, connected affine algebraic group, its character sheaves are exactly the multiplicative local systems on $\mathbb{T}_r$. With Lusztig's result that parabolic induction takes character sheaves to character sheaves, a natural first step towards a (at least *generic*) theory of character sheaves is an affirmative answer to the following conjecture:

Conjecture 3.18 (Lusztig, 2006)**.** *Let $\mathcal{L}$ be a multiplicative local system on $\mathbb{T}_r$ which is $(\mathfrak{t}, \mathcal{L}_\psi)$-equivariant for a weakly $(\mathbf{T}, \mathbf{G})$-generic $\mathcal{L}_\psi$. Then $\mathrm{pInd}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L})$ is perverse.*

Recall that the proof of Theorem 3.8 in the $r = 0$ setting relied completely two characteristics of π: it is proper and semismall. The essential difficulty in establishing Conjecture 3.18 is that for $r > 0$, both of these properties fail for π. Nevertheless, this difficulty was overcome and Conjecture 3.18 was established by Lusztig in the following cases:

- $\mathbb{G}_r$ is the jet scheme of GL_2 and $r = 1$ [Lus06]
- $\mathbb{G}_r$ is the jet scheme of a connected reductive group and $r = 1, 3$ [Lus17]

In [BC24], Bezrukvanikov and I develop an completely different approach. Our framework, which also works for general positive-depth parabolic induction (from a Levi, not just from a torus), resolves Conjecture 3.18.

Theorem 3.19 (Bezrukavnikov–C., 2024)**.** *Conjecture 3.18 is true. Moreover, if $\mathcal{L}_\psi$ is $(\mathbf{T}, \mathbf{G})$-generic, then $\mathrm{pInd}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L})$ is simple.*

In fact, the compatibility between $\mathrm{pInd}_{\mathbb{T}_r}^{\mathbb{G}_r}$ and $R_{\mathbb{T}_r}^{\mathbb{G}_r}$ also holds, just as (5) illustrates for the $r = 0$ setting.

Theorem 3.20 (Bezrukavnikov–C., 2024)**.** *Assume $q \gg 0$. Let $\mathcal{L}$ be $(\mathfrak{t}, \mathcal{L}_\psi)$-equivariant for a weakly $(\mathbf{T}, \mathbf{G})$-generic $\mathcal{L}_\psi$. If $\mathcal{L}$ is σ-equivariant with associated $\bar{T}_r$-character θ, then*

$$\Theta_{\mathrm{pInd}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L}), \sigma} = (-1)^{\dim \mathbb{T}_r} \cdot \Theta_{R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)}.$$

We remark that the largeness assumption on q is very mild: we need only guarantee that $\bar{T}_0$ has a regular element.

3.4. **Generic subcategories.** The two notions of weak $(\mathbf{M}, \mathbf{G})$-genericity and $(\mathbf{M}, \mathbf{G})$-genericity as in Definition 2.8 have sheaf-theoretic analogues, in the spirit of Definition 3.17. In [BC24], we define full subcategories of $\mathcal{D}_{\mathbb{M}_r}(\mathbb{M}_r)$ and $\mathcal{D}_{\mathbb{G}_r}(\mathbb{G}_r)$ associated to each weakly $(\mathbf{M}, \mathbf{G})$-generic $\mathcal{L}_\psi$. For the purpose of these notes, we will use a less intrinsic definition of these generic subcategories.

Definition 3.21 (generic subcategory)**.** Let $\mathcal{L}_\psi$ be a weakly $(\mathbf{M}, \mathbf{G})$-generic multiplicative local system on $\mathfrak{m} := \ker(\mathbb{M}_r \to \mathbb{M}_{r-1})$. We define the two associated generic subcategories $\mathcal{D}^\psi_{\mathbb{M}_r}(\mathbb{M}_r)$ and $\mathcal{D}^\psi_{\mathbb{G}_r}(\mathbb{G}_r)$ in the following way: $\mathcal{D}^\psi_{\mathbb{M}_r}(\mathbb{M}_r)$ to be the full subcategory of $\mathcal{D}_{\mathbb{M}_r}(\mathbb{M}_r)$ consisting of $(\mathfrak{m}, \mathcal{L}_\psi)$-equivariant complexes and $\mathcal{D}^\psi_{\mathbb{G}_r}(\mathbb{G}_r)$ is its image under parabolic induction.

Note that in general it is not true that the image of a functor is a category. However, in the present setting, this is not a concern as it turns out that the parabolic induction functor $\mathrm{pInd}_{\mathbb{M}_r}^{\mathbb{G}_r}$ is fully faithful on $\mathcal{D}^\psi_{\mathbb{M}_r}(\mathbb{M}_r)$. In [BC24], we prove:

Theorem 3.22. *If $\mathcal{L}_\psi$ is weakly $(\mathbf{M}, \mathbf{G})$-generic, then $\mathrm{pInd}_{\mathbb{M}_r}^{\mathbb{G}_r}$ is t-exact on $\mathcal{D}^\psi_{\mathbb{M}_r}(\mathbb{M}_r)$. If $\mathcal{L}_\psi$ is $(\mathbf{M}, \mathbf{G})$-generic, then $\mathrm{pInd}_{\mathbb{M}_r}^{\mathbb{G}_r}$ furthermore defines an equivalence of categories*

$$\mathcal{D}^\psi_{\mathbb{M}_r}(\mathbb{M}_r) \to \mathcal{D}^\psi_{\mathbb{G}_r}(\mathbb{G}_r).$$

Note that Theorem 3.19 follows immediately from Theorem 3.22 in the case $\mathbf{M} = \mathbf{T}$.

An important corollary of Theorem 3.22 is that generic parabolic induction can be iterated! This is decidedly inspired from the structure of Yu's and Kim–Yu's construction of types for representations of p-adic groups.

Definition 3.23. A *clipped generic datum*[1] is a tuple $\ddot{\Psi} := (\vec{\mathbf{G}}, \mathbf{x}, \vec{r}, \vec{\mathcal{L}})$ satisfying:

- **D0** $\mathbf{T}$ is an unramified maximal torus of $\mathbf{G}$
- **D1** $\vec{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \ldots, \mathbf{G}^d)$ is a strictly increasing sequence of twisted Levi subgroups of $\mathbf{G}$ which contain $\mathbf{T}$; we assume $\mathbf{G}^d = \mathbf{G}$
- **D2** $\mathbf{x}$ is a point in the apartment of $\mathbf{T}$ in the building of $\mathbf{G}$
- **D3** $\vec{r} = (r_0, r_1, \ldots, r_d)$ is a sequence of integers satisfying $0 < r_0 < r_1 < \ldots < r_{d-1} \leq r_d$ if $d > 0$ and $0 \leq r_0$ if $d = 0$.
- **D4** $\vec{\mathcal{L}} = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_d)$ is a sequence where for $0 \leq i \leq d$, $\mathcal{L}_i$ is a multiplicative local system on $\mathbb{G}^i_{r_i}$ which is $(\mathfrak{g}^i, \mathcal{L}_{\psi_i})$-equivariant for a $(\mathbf{G}^i, \mathbf{G}^{i+1})$-generic $\mathcal{L}_{\psi_i}$; if $r_{d-1} = r_d$, we assume $\mathcal{L}_d$ is the constant local system.

By iterating the parabolic induction functor $\mathrm{pInd}_{\mathbb{G}^i_{r_i}}^{\mathbb{G}^{i+1}_{r_i}}$ taken on the generic subcategories $\mathcal{D}^{\psi_i}_{\mathbb{G}^i_{r_i}}(\mathbb{G}^i_{r_i}) \to \mathcal{D}^{\psi_i}_{\mathbb{G}^{i+1}_{r_i}}(\mathbb{G}^{i+1}_{r_i})$, we obtain the following: For any clipped generic datum $\ddot{\Psi} := (\vec{\mathbf{G}}, \mathbf{x}, \vec{r}, \vec{\mathcal{L}})$, we have a functor

$$\mathrm{pInd}_{\ddot{\Psi}} \colon \mathcal{D}_{\mathbb{G}^0_0}(\mathbb{G}^0_0) \to \mathcal{D}_{\mathbb{G}_r}(\mathbb{G}_r)$$

defined by

$$\mathcal{K} \mapsto (\pi^d)^\dagger \,\mathrm{pInd}_{\mathbb{G}^{d-1}_{r_{d-1}}}^{\mathbb{G}^d_{r_{d-1}}} \left((\pi^{d-1})^\dagger \,\mathrm{pInd}_{\mathbb{G}^{d-2}_{r_{d-2}}}^{\mathbb{G}^{d-1}_{r_{d-2}}} \left(\cdots (\pi^1)^\dagger \,\mathrm{pInd}_{\mathbb{G}^0_{r_0}}^{\mathbb{G}^1_{r_0}} ((\pi^0)^\dagger \mathcal{K} \otimes \mathcal{L}_0) \otimes \mathcal{L}_1 \cdots \right) \otimes \mathcal{L}_{d-1} \right) \otimes \mathcal{L}_d,$$

where $\pi^i \colon \mathbb{G}^i_{r_i} \to \mathbb{G}^i_{r_{i-1}}$ for each i and each $\dagger$ superscript denotes smooth pullback. By Theorem 3.22, we obtain that:

[1]This terminology was first used in [CO23] in the context of representation theory.

Corollary 3.24. $\mathrm{pInd}_{\ddot{\Psi}}$ *is t-exact and fully faithful. In particular, if* $\mathcal{K}$ *is an irreducible character sheaf* $\mathbb{G}_0^0$*, then* $\mathrm{pInd}_{\ddot{\Psi}}(\mathcal{K})$ *is an irreducible character sheaf on* $\mathbb{G}_r$*.*

By [Lus06], the σ-equivariant character sheaves on $\mathbb{T}_r$ are exactly the multiplicative local systems $\mathcal{L}_\theta$ associated to a character θ of $\bar{T}_r$. Let us assume that θ *has a Howe factorization* in the sense of Kaletha [Kal19, Definition 3.6.2] (see the paragraph preceding Theorem 2.9 for a brief description). It is a theorem of Kaletha that this is automatic if p is sufficiently large. Converting all characters to multiplicative local systems, we exactly get a sequence of σ-equivariant local systems $\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_d$ on a sequence $\mathbb{T}_0, \mathbb{G}^0_{r_0}, \mathbb{G}^1_{r_1}, \ldots, \mathbb{G}^d_{r_d}$ of algebraic groups. Moreover, forgetting $\mathcal{L}_{-1}$ and $\mathbb{T}_0$, this data satisfies Definition 3.23; denote the induced truncated generic datum by $\ddot{\Psi}_{\theta_+} = (\vec{\mathbf{G}}, \mathbf{x}, \vec{r}, \vec{\mathcal{L}}_{\theta_+})$. Therefore, Corollary 3.24 yields an association

$$\mathcal{L}_\theta \mapsto \mathcal{F}_\theta := \mathrm{pInd}_{\ddot{\Psi}_{\theta_+}}(\mathrm{pInd}_{\mathbb{T}_0}^{\mathbb{G}_0^0}(\mathcal{L}_{-1}))$$

satisfying:

(1) $\mathcal{F}_\theta$ is a perverse sheaf since $\mathrm{pInd}_{\mathbb{T}_0}^{\mathbb{G}_0^0}(\mathcal{L}_{-1})$ is a perverse sheaf by Theorem 3.8.
(2) If $\mathcal{L}_{-1}$ has trivial stabilizer in $W_{\mathbb{G}_0^0(\overline{\mathbb{F}}_q)}(\mathbb{T}_0)$, then $\mathcal{F}_\theta$ is simple.

Combining (2) with Kaletha's theory of Howe factorizations developed in [Kal19], we then have:

Theorem 3.25. *For any* $\bar{T}_r$*-character* θ *with Howe factorization, we have an associated* $\mathbb{G}_r$*-equivariant perverse sheaf* $\mathcal{F}_\theta$*. Moreover, if* θ *has trivial* $W_{\mathbb{G}_r(\overline{\mathbb{F}}_q)}(\mathbb{T}_r)$*-stabilizer, then* $\mathcal{F}_\theta$ *is simple.*

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Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48105, USA.

Email address: charchan@umich.edu