Chan Lecture 4: Character sheaves

So far:

$$\begin{array}{ccc} & X & \\ G \circlearrowright & & \circlearrowleft T \end{array} \rightsquigarrow H_c^*(X)$$

$(G\times T)$-repn

Another way:

$\sigma \circlearrowright$ ~~X~~ ~~ft~~ sch $/ \overline{\mathbb{F}}_q \circlearrowleft \sigma$

$\mathcal{F}$ = complex of constr $\overline{\mathbb{Q}}_\ell$ sh on X

$\rightsquigarrow$ for each closed pt $x \in \overline{X}$, geom Frob σ_x acts on the stalk $\underbrace{\mathcal{F}_x}_{\text{v.sp.}}$.

$\Theta_{\mathcal{F},\sigma} : \overline{X} \longrightarrow \overline{\mathbb{Q}}_\ell,\ x \mapsto$ ~~Σ~~ $\mathrm{Tr}(\sigma_x ; H_c^*(\mathcal{F}_{\bar{x}}))$.

sheaf	function
skyscraper sh.	delta fn
$\mathcal{L}_\theta$ on $\mathbb{T}$ mult loc sys	mult character $\theta: \mathbb{T} \to \mathbb{C}^*$
pullback	pullback
pushforward	average (sum)
base-change	change-of-var
proj. form.	factoring out
FT	FT
convolution	convolution

Char sh on $\mathbb{G}$

The Grothendieck Springer resolution:

$$\begin{tikzcd}
& \tilde{\mathbb{G}} := \{ (g, h\mathbb{B}) \in \mathbb{G} \times \mathbb{G}/\mathbb{B} : h^{-1}gh \in \mathbb{B} \} \arrow[dl, "f"'] \arrow[dr, "\pi"] & \\
\mathbb{T} & & \mathbb{G}
\end{tikzcd}$$

$$\mathrm{pr}(h^{-1}gh) \in \mathbb{T} \qquad g \in \mathbb{G}$$

Def. (geom parabolic induction).

$$\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}) := \pi_! f^* \mathcal{L}$$

Ex: $\sigma \circlearrowright \mathbb{G}, \mathbb{T}, \mathbb{B}$.

Then

$$4_{\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}_\theta)}(g) = \sum_{\substack{h\bar{\mathbb{B}} \in \bar{\mathbb{G}}/\bar{\mathbb{B}} \\ \text{s.t. } h^{-1}gh \in \bar{\mathbb{B}}}} \theta(\mathrm{pr}(h^{-1}gh)) = 4_{\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\theta)}(g).$$

$\sigma \curvearrowright \mathbb{T} \hookrightarrow \mathbb{G} \curvearrowleft \sigma$ 4

$\mathbb{B}$ not nec. σ-stable

__Thm.__ (Lusztig) $\theta : \overline{T} \to \mathbb{C}^\times$

a) $\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}_\theta)$ ← first ex of char sh.

(1985) is a (simple) perverse sheaf (if θ is regular).

b) $④\,\mathrm{pInd}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}_\theta) = \bullet\ ④\, R_{\mathbb{T}}^{\mathbb{G}}(\theta)$

(1990)

__2004__: ~~For~~ On What other gps could there be a theory of character sheaves?

- ~~charac~~ unipotent (Boyarchenko, Boyarchenko–Drinfeld)
- $\mathbb{G}_r$, at least "generically" (resolved by Lusztig for $r=1$, $r=3$)

Generic char. sh on $\mathbb{G}_r$

$$\widetilde{\mathbb{G}_r} := \{(g, h\mathbb{B}_r) \in \mathbb{G}_r \times \mathbb{G}_r/\mathbb{B}_r : h^{-1}gh \in \mathbb{B}_r\}$$

$$\begin{tikzcd} & \widetilde{\mathbb{G}_r} \arrow[dl, "f"'] \arrow[dr, "\pi^*"] & \\ \mathbb{T}_r & & \mathbb{G}_r \end{tikzcd}$$

<u>Def</u>. (geom posdepth par ind) $p\mathrm{Ind}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L}) := \pi_! f^* \mathcal{L}_\theta$

<u>Conj</u>. $\theta : \overline{T}_r \to \mathbb{C}^\times$ which is $(\underline{I}, \underline{G})$-gen

(Lusztig) (Fintzen's talk

Then $p\mathrm{Ind}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L}_\theta)$ is simple perverse.

<u>ThmA</u> (Bezrukavnikov–C.)

Truth of Conj. 2024

<u>ThmB</u> (——— $\parallel$ ———)

$$ⓗ\,\mathrm{pInd}^{G_r}_{\Pi_r}(\mathcal{L}_\theta) = \bullet\; ⓗ\,R^{G_r}_{\Pi_r}(\theta).$$

~~Takeaway~~: It's easier to prove

<u>Strat of pf</u>: props of $\mathrm{pInd}^{G_r}_{\Pi_r}$ than

~~plus~~ to prove $\mathrm{pInd}^{G_r}_{\Pi_r}(\mathcal{L})$ sat. props.

Show $\mathrm{pInd}^{G_r}_{\Pi_r}(\mathcal{L}_\theta)$ ~~perser~~ by showing

$\mathrm{pInd}^{G_r}_{\Pi_r}$ is t-exact (on some gen. subcat $\mathcal{C}$)

(geom) par. Ind. $\updownarrow$ s.c. (rational)

$\mathrm{pInd}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L}_\theta)$

④ $\updownarrow$ Thm B

$R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$

cInd $\updownarrow$ Thm (C-Oi)

$\pi^{\mathrm{alg} \times \mathrm{FKS}}_{(T,\theta)}$

~~reg. s.c. rep.~~

$\underline{T} \hookrightarrow \underline{G}$ ellipt max tor, unram.

!!!

Generic subcats.

Def. $\psi : t \to \mathbb{C}^\times$, $(\underline{T}, \underline{G})$-generic.

$D^{\psi}_{\mathbb{T}_r}(\mathbb{T}_r) :=$ subcat consisty of objects which are (t, ψ)-equiv wrt to the mult action of t on $\mathbb{T}_r$

$$t = \ker(\mathbb{T}_r \longrightarrow \mathbb{T}_{r-1})$$

$D^{\psi}_{\mathbb{G}_r}(\mathbb{G}_r) =$ "averaged" vers in $\mathbb{G}_r$.

Thm. $\mathrm{pInd}_{\mathbb{T}_r}^{\mathbb{G}_r} : D^{\psi}_{\mathbb{T}_r}(\mathbb{T}_r) \to D^{\psi}_{\mathbb{G}_r}(\mathbb{G}_r)$

is a t-exact equiv of categories.

Character sh on $\mathbb{G}$.

Observation:

- we can also define for ψ $(\underline{G}', \underline{G})$-gen.

- $$D^{\psi}_{\mathbb{G}_r'}(\mathbb{G}_r') \xrightarrow[\mathrm{pInd}^{\mathbb{G}_r}_{\mathbb{G}_r'}]{} D^{\psi}_{\mathbb{G}_r}(\mathbb{G}_r)$$

  is a t-exact equiv of cats.

Yu datum:

① $\underline{G}^0 \subsetneq \underline{G}^1 \subsetneq \cdots \subset \underline{G}^n = \underline{G}$

② x vertex in $\mathcal{B}(\underline{G}^0, F)$

③ ρ cuspidal irrep of $G^0_x / G^0_{x,0+}$

④ $0 < r_0 < r_1 < \cdots \leq r_n$ rat'l #s.

⑤ $\phi_0 \quad \phi_1 \quad \cdots \quad \phi_n$

geom Yu datum:

① $\underline{G}^0 \subsetneq \underline{G}^1 \subsetneq \cdots \subset \underline{G}^n$ genuine Levi over F^{ur}

② x

③ $\mathcal{F}$ char sh on $\mathbb{G}_0^0$ ← conn red alg gp

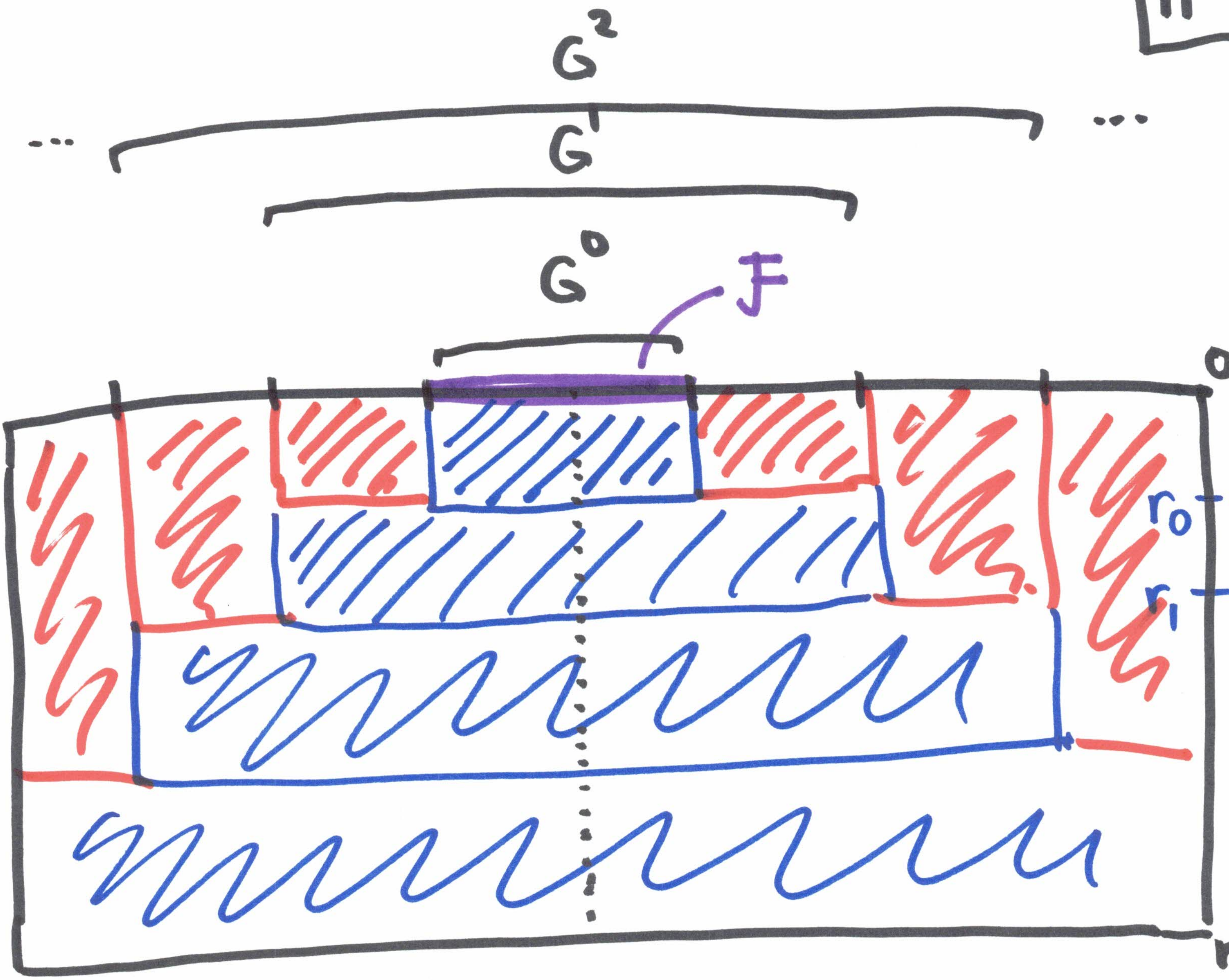
④ $r_0 < r_1 < \cdots \leq r_n$ pos int.

④.5 $\mathbb{G}_{r_0}^0 \quad \mathbb{G}_{r_1}^1 \quad \cdots \quad \mathbb{G}_{r_n}^n$

⑤ $\mathcal{L}_0 \quad \mathcal{L}_1 \quad \cdots \quad \mathcal{L}_n$

⇝ successively apply "generic ~~par~~ geom parab. ind" to obtain char. sheaf on $\mathbb{G}_{r_n}^n$.

$r_n = r$ $\qquad \| \; \mathbb{G}_r^n$



<u>In particular</u>: $\theta : \overline{T}_r \to \mathbb{C}^\times$ w/ Howefact

$\rightsquigarrow$ get a datum, $F = p\mathrm{Ind}_{\mathbb{T}}^{\mathbb{G}^0}(\mathcal{L}_{\phi^{-1}})$.

$\rightsquigarrow$ run the algorithm & get a char sh (simple, perverse) on $\mathbb{G}_r$.

Thm (C-Oi)

$$④_{Q^{G_r}_{\Pi_r}(\theta_+)} = FT\left(\delta_{\cancel{x}\,\text{orbit}}\right).$$

(get a pos-depth Springer hypoth.)