Chan Lecture 3: Very regular elements

Def. A reg s.s. elt $\gamma \in G$ is called **tame very regular** if

- the conn centr $\underline{T}_\gamma$ of γ in $\underline{G}$ is a tamely ram max'l tor.
- $\alpha(\gamma) \not\equiv 1 \mod \mathfrak{p}_{\breve{F}}$ $\forall \alpha$ root of $\underline{T}_\gamma$ in $\underline{G}$.

Ex. $\underline{G} = GL_2$

- $\underline{T}$ unram, elliptic. $T \cong L^\times$, L/F deg 2 unram extn

  Then $\gamma \in L^\times$ is reg ss if $\gamma \in L^\times \setminus F^\times$.

  $\gamma \in \mathcal{O}_L^\times$ is very reg if $\bar{\gamma} \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times$

- $\underline{T}$ ram, elliptic, $T \cong E^\times$, E/F deg 2 ram extn

  Then $\gamma \in E^\times$ is very reg if val(γ) is odd.

Thm (C-01, 2025) (MAIN)

Assume $q \gg 0$. $(\underline{T}, \theta)$ ← reg. Then there exists

↑ unram elliptic

at most one irrep π of $G_{x,0}$ s.t.

$$(*) \quad ④_\pi(\gamma) = \pm 1 \cdot \sum_{w \in W_{G_{x,0}}(\underline{T}_\gamma, \underline{T})} \theta^w(\gamma) \quad \forall \text{ tame very reg. } \gamma.$$

$\bar{G}$

Thm. (———) Assume $q \gg 0$, $(\mathbb{T} \hookrightarrow \mathbb{G}, \theta)$.

allowed to be split; c reg

There exists ~~at most~~ EXACTLY one irrep of $\mathbb{G}$ s.t.

$$④_\pi(\gamma) = \cdot \sum_{w \in W} \theta^w(\gamma) \quad \forall \text{ rss } \gamma.$$

$$\left(\pi = R_{\mathbb{T}}^{\mathbb{G}}(\theta)\right).$$

Recall:

Thm. (Deligne–Lusztig char formula)

$$\Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}(su) = \frac{1}{|\bar{Z}^{\circ}_{\mathbb{G}}(s)|} \sum_{\substack{g \in \bar{G} \\ \text{s.t. } gsg^{-1} \in \bar{T}}} \theta^{g}(s) \cdot \Theta_{Q_{\mathbb{T}^{g}}^{Z^{\circ}_{\mathbb{G}}(s)}}(u) \quad (1).$$

Cor. s reg. ss., $u = 1$.

$$\Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}(s) = \sum_{w \in W_{\bar{G}}(\mathbb{T})} \theta^{w}(s).$$

Ex. consider $\bar{T}, \bar{T}'$ in $GL_2\mathbb{F}_q$, θ, θ' reg.

• $\mathrm{Ind}_{\bar{B}}^{\bar{G}}(\theta)$ is the unique irrep of $\bar{G}$ s.t.

$$④(\gamma) = \begin{cases} \theta\begin{pmatrix} a & \\ & b\end{pmatrix} + \theta\begin{pmatrix} b & \\ & a\end{pmatrix} & \text{for } \gamma \sim \begin{pmatrix} a & \\ & b\end{pmatrix},\ a \neq b \\ 0 & \text{for } \gamma \text{ dist. eig vals} \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times \end{cases}$$

$q^2 - 3q + 2 > 4$

• $R_{\mathbb{T}'}^{\mathbb{G}}(\theta')$ is the unique irrep of $\bar{G}$ s.t.

$$④(\gamma) = \begin{cases} 0 & \text{for } \gamma \sim \begin{pmatrix} a & \\ & b\end{pmatrix},\ a \neq b \\ \theta(\gamma) + \theta(\gamma^q) & \text{for } \gamma \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times. \end{cases}$$

$q^2 - q > 4$

Can use the $\tilde{G}$ "litmus test" thm to give ~~characte~~ "litmus test" results for certain s.c. reps of p-adics.

2 results.

Thm (C-Oi, 2023) $(\underline{I} \subset \underline{G}, \theta)$ tame ell. reg par.

If T has enough very regular elts, then the associated (FKS-twisted) reg s.c. is the ~~characterize~~ unique s.c. with char.

$$④(\gamma) = \bullet \sum_{w} \cdots \theta^{w}(\gamma) \quad \gamma \text{ vreg.}$$

$q \gg 0$

<u>Thm</u>. π s.c. is unip.

$\Leftrightarrow$ (i) $④_\pi \big|_{T_{vreg}}$ constant for any T.

(i) $④_\pi \big|_{T_{vreg}} \neq 0$ for a max'ly unram ell max'l tor T.

Pf of main litmus test.

Assume π, π' are smooth irreps of $G_{x,0}$ sat. (*)
(note: $\bullet$, $\bullet'$ need not be the same a priori)

Goal: $\langle \pi, \pi' \rangle \neq 0$.

Now: $\langle \pi, \pi' \rangle = \langle \pi, \pi' \rangle_{vreg} + \langle \pi, \pi' \rangle_{nvrg}$

Cauchy-Schwarz:

$$\left| \langle \pi, \pi' \rangle_{nvreg} \right| \leq \langle \pi, \pi \rangle_{nvreg}^{1/2} \cdot \langle \pi', \pi' \rangle_{nvreg}^{1/2}$$

$$= \langle \pi, \pi \rangle_{nvreg}.$$

(By assump: $\langle \pi, \pi \rangle = 1 = \langle \pi, \pi \rangle_{vreg} + \langle \pi, \pi \rangle$
|| || $\Rightarrow$ || not
$\langle \pi', \pi' \rangle = 1 = \langle \pi', \pi' \rangle_{vrg} + \langle \pi', \pi' \rangle$)

ETS: $\langle \pi, \pi \rangle_{nvreg} < \frac{1}{2}$.

ETS: $\langle \pi, \pi \rangle_{vreg} > \frac{1}{2}$.

Compute:

$$\langle \pi, \pi \rangle_{vreg} = \frac{1}{|G_{x,0}|} \sum_{\gamma \in (G_{x,0})_{vreg}} \sum_{w,w' \in W(T_\gamma, T)} \theta^{w}(\gamma) \cdot \overline{\theta^{w'}(\gamma)}$$

$$= \frac{1}{\cancel{|G_{x,0}|}} \cdot \frac{\cancel{|G_{x,0}|}}{|N(T_\gamma, T)|} \sum_{t \in T_{vreg}} \sum_{w,w'} \theta^{w}(t) \cdot \overline{\theta^{w'}(t)}$$

$$= \frac{1}{|N(T_\gamma, T)|} \sum_{w,w'} \left(|T| \cdot \langle \theta^{w}, \theta^{w'} \rangle - \sum_{t \in T_{nvreg}} \theta^{w}(t) \cdot \overline{\theta^{w'}(t)} \right)$$

$$\geq \frac{1}{|N(T_\gamma, T)|} \cdot \sum_{w,w'} \left(|T| \cdot \langle \theta^{w}, \theta^{w'} \rangle - |T_{nvreg}| \right)$$

$$= 1 - \frac{|T_{nvreg}|}{|T_0|} \cdot |W| \overset{?}{>} \frac{1}{2}$$

This is $> \frac{1}{2}$

$$\Leftrightarrow \frac{|\bar{T}_0|}{|\bar{T}_{nvrag}|} > 2|W|. \qquad \square$$

Now we know $\exists$ at most one irrep of $G_{x.0}$ sat. $(*)$.

Recall:

Thm. (pos depth DL char formula)

$$\Theta_{R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)}(su) = \frac{1}{|Z^0_{\mathbb{G}_r}(s)|} \sum_{\substack{g \in \overline{\mathbb{G}}_r \\ \text{s.t. } gsg^{-1} \in \overline{\mathbb{T}}_r}} \theta^g(s) \cdot \Theta_{Q_{\mathbb{T}_r^g}^{Z^0_{\mathbb{G}_r}(s)}(\theta_+)}(u)$$

Cor. su very regular.

$$\Theta_{R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)}(su) = \sum_{w \in W_{\overline{\mathbb{G}}_r}(\mathbb{T}_r)} \underbrace{\theta^w(s) \cdot \theta_+^w(u)}_{\theta^w(su)}$$

$\therefore \exists!$ irrep of $G_{x,0}$ sat $(*)$.

Implication for theory of p-adic grps: (inserted under "for": rep)

In Tasho's lecture:

$$(\underline{T} \subset \underline{G}, \theta) \longmapsto \pi^{alg}_{(\underline{T} \subset \underline{G}, \theta)}$$

$$(\underline{T} \subset \underline{G}, \theta) \mapsto (\underline{T} \subset \underline{G}, \theta \cdot \xi) \mapsto \pi^{alg \times FKS}_{(\underline{T} \subset \underline{G}, \theta)}$$

Q: How is $R^{G_r}_{T_r}(\theta)$ related?

unram

$\underline{T} \subset \underline{G}$ ⟿ a point $x \in \mathcal{B}(G)$ ⟷ parahoric $G_{x,0}$

$\theta: T \to \mathbb{C}^\times$ ⟿ depth $r \geq 0$ ⟷ a Moy-Prasad filtr. subgp $G_{x,r+}$

One can constr. an alg. gp $\mathbb{G}_r / \overline{\mathbb{F}}_q \circlearrowleft \sigma$

and $\mathbb{T}_r, \mathbb{B}_r, \mathbb{U}_r$. s.t.

$$\underset{\sigma}{\circlearrowleft} \qquad \overline{G}_r = G_{x,0} / G_{x,r+}$$

$$\overline{T}_r = \text{subquot of } T$$

$\rightsquigarrow$ have pos. depth DL var

$$H_c^*(X_{\mathbb{T}_r \subset \mathbb{G}_r})_\theta$$

$$\underset{G_{x,0}}{\circlearrowright} \qquad \underset{T_0}{\circlearrowleft}$$

$$\rightsquigarrow (\underline{T} \subset \underline{G}, \theta) \mapsto \pi^{\text{geom}}_{(T,\theta)} := \text{c-Ind}_{ZG_{x,0}}^{G} (R^{\mathbb{G}_r}_{\mathbb{T}_r}(\theta))$$

$$(I \subset \underline{G}, \theta) \mapsto \pi^{alg}_{(T,\theta)} \not\cong$$

$$\mapsto \pi^{alg \times FKS}_{(T,\theta)}$$

$$\mapsto \pi^{geom}_{(T,\theta)}$$

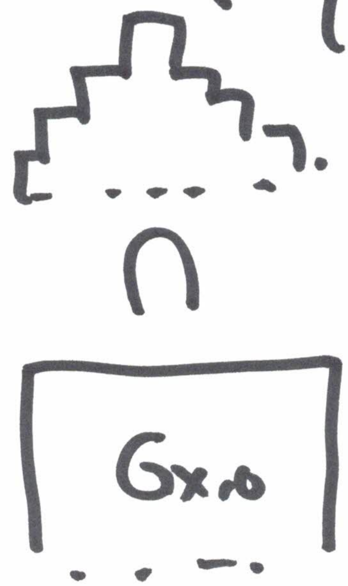
<u>Thm.</u> (C-Oi) θ reg., $p \gg 0$, $q \gg 0$.

$$\pi^{geom}_{(T,\theta)} \cong \pi^{alg}_{(T,\theta\cdot\xi)} \cong \pi^{alg \times FKS}_{(T,\theta)}.$$

(yay: $\pi^{geom}_{(T,\theta)}$ respects Langlands phenomena.).

Pf:

$$\pi^{alg \times FKS}_{(T,\theta)} = c\mathrm{Ind}^{G}\left(\tau^{FKS}_{(T,\theta)}\right)$$



$$= c\mathrm{Ind}^{G}\left(\mathrm{Ind}^{G_{x,0}}\left(\tau^{FKS}_{(T,\theta)}\right)\right).$$

Prop: (L1) $(\gamma) = \bullet \sum_{w \in W(T_\gamma, T)} \theta^w(\gamma).$

$\Rightarrow$ apply litmus test. $\forall\ \gamma$ vrg. $\square$