lecture 1: Deligne-Lusztig theory

lecture 2: Lusztig's conjecture and positive-depth Deligne-Lusztig varieties

lecture 3: ~~very~~ very regular elements

lecture 4: Character sheaves

Chan Lecture 2:
Lusztig's Conjecture and positive-depth Deligne-Lusztig varieties

$\underline{G}$ = conn red / F, $\underline{T}$ = max'l torus, elliptic, unram, / F

$$\begin{array}{ccc} \underline{B} & \subset & \underline{G} \\ \cup & \supset & \\ \underline{T} & & \underline{U} \end{array}$$

Borel, F^{ur}-rat'l

Conj. (Lusztig, 1979)

$$X_\infty := \{ g \in \underline{G}(F^{ur}) : g^{-1}\sigma(g) \in \underline{U}(F^{ur}) \}$$

$G \circlearrowright X_\infty \circlearrowleft T$

$\underline{U}(F^{ur}) \cap \sigma^{-1}(\underline{U}(F^{ur}))$

- should be an ind-scheme over $\overline{\mathbb{F}}_q$
- should have homology gps $H_i(X_\infty)$

History:

- 1979, Lusztig: $\mathcal{D}^1_{1/n}$
- 2012, Boyarchenko: $\mathcal{D}^\times_{1/n}$ + $H^i(X_\infty)_\theta$ for θ smallest pos depth.
- 2016, Boyarchenko-Weinstein:

  piece of $X_\infty \longleftrightarrow$ special affinoid in the Lubin Tate tower (Weinstein, Imai-Tsushima, Mieda, Tokimoto, ...)
- 2016-2020; Chan: completed the comp. of $H^i(X_\infty)_\theta$ for arb θ.
- 2021-2023, Chan-Ivanov: any inn fm of GL_n.

  + found an "ADLV at inf level" that's isom to X_∞

- 2016: Conj. ~~pf~~ Fargues
- 2023, Takamatsu: X_∞ & ADLV for GSp_{2n}.
- 2022-2023, Ivanov: X_∞ is an ind sch.
  Ivanov-Nie
  If $\underline{T}$ is Coxeter, then one has a decomp. $X_\infty = \bigsqcup \underbrace{X_\infty^\circ}_{\text{inf. sch. "bdd part" "parahoric part".}}$
- future, Ivanov: homology?

Example: X_∞ in the case $GL_2 = \underline{G}$.

Set-up: $\sigma: GL_2 \to GL_2$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sigma(d) & \sigma(c) \\ \sigma(b) & \sigma(a) \end{pmatrix}$

$T = L^\times$, L deg 2 unram extn of F $\leftarrow$ $\underline{T} \hookrightarrow GL_2$, diag.

$\underline{B} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. $\underline{U} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Then:

$$X_\infty = \left\{ g \in GL_2(F^{ur}) : g^{-1}\sigma(g) \in \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

$$\Updownarrow$$

$$\sigma(g) \in g \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} \sigma(d) & \sigma(c) \\ \sigma(b) & \sigma(a) \end{pmatrix} \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

$$a = \sigma(d)$$
$$c = \sigma(b)$$
$$\det \in F^\times.$$

$$= \left\{ \begin{pmatrix} \sigma(d) & b \\ \sigma(b) & d \end{pmatrix} \in GL_2(F^{ur}) : \det \in F^\times \right\}.$$

Rmk. For $\overline{\mathbb{F}}_q$ instead:

$$X_{T' \subset G} = \left\{ \begin{pmatrix} d^q & b \\ b^q & d \end{pmatrix} \in GL_2(\overline{\mathbb{F}}_q) : \det \in \mathbb{F}_q^\times \right\}$$

$$= \mathbb{V}\left(\left(-b^{q+1} + d^{q+1}\right)^{q-1} = 1 \right).$$

Thm. $$X_\infty = \bigsqcup_{\gamma \in GL_2(F)/GL_2(\mathcal{O}_F)} \gamma \cdot X_\infty^\circ$$

where $$X_\infty^\circ = \left\{ \begin{pmatrix} \sigma(d) & \varpi b \\ \sigma(b) & d \end{pmatrix} \in GL_2(\mathcal{O}_{F^{ur}}) : \det \in \mathcal{O}_F^\times \right\}.$$

and $$X_\infty^\circ = \varprojlim_r X_r^\circ$$

where $$X_r^\circ = \left\{ \begin{pmatrix} & \\ & \end{pmatrix} \in GL_2(\mathcal{O}_{F^{ur}}/\varpi^{r+1}) : \det \in (\mathcal{O}_F/\varpi^{r+1})^\times \right\}.$$

Here: $$\begin{array}{ccccc} X_\infty^\circ & , & & X_r^\circ & \circlearrowleft (\mathcal{O}_D/\varpi^{r+1})^\times \\ \circlearrowright & \circlearrowleft & & \circlearrowright & \\ GL_2(\mathcal{O}_F) & T_0 = \mathcal{O}_D^\times & & GL_2(\mathcal{O}_F/\varpi^{r+1}) & \end{array}$$

It turns out: $X_r^\circ \longrightarrow X_{r-1}^\circ$

$X_r^\circ / (1+\varpi^r) \,\big/\, (1+\varpi^{r+1})$

$\mathbb{F}_{q^2} \cong (1+\varpi^r)/(1+\varpi^{r+1})$

$\Rightarrow H_i(X_r^\circ) = H_i(X_{r-1}^\circ)$

$\Rightarrow H_i(X_{r-1}^\circ) \hookrightarrow H_i(X_r^\circ)$

$\Rightarrow H_i(X_\infty^\circ) := \varinjlim_r H_i(X_r^\circ)$

Thm (C.-Ivanov) $\theta : T \to \mathbb{C}^\times$ depth r.

Then $H_*(X_\infty)_\theta = \operatorname{c-Ind}_{Z(F)GL_2(\mathcal{O}_F)}^{GL_2(F)} (H_*(X_r)_\theta)$

* Special case: say θ has depth 0.

Then $H_*(X_\infty)_\theta = \operatorname{c-Ind}_{Z(F)GL_2(\mathcal{O}_F)}^{GL_2(F)} (R_{\mathbb{T}}^{\mathbb{G}}(\theta)$

These X_r° are generalizations of DL varieties.

Positive-depth Deligne-Lusztig varieties.

Jet scheme setting : $\mathbb{G}$ conn red / $\overline{\mathbb{F}}_q \circlearrowleft \sigma$

$$\begin{array}{l} \mathbb{T} \hookrightarrow \mathbb{G} \quad \sigma\text{-stable max'l torus} \\ \cap \\ \mathbb{B} \supset \mathbb{U} \end{array}$$

$\mathbb{G}_r$ = rth jet scheme for $\mathbb{G}$

$$A \mapsto \mathbb{G}(A[t]/t^{r+1}).$$

$$\overline{G}_r := \mathbb{G}_r(\overline{\mathbb{F}}_q)^\sigma$$

__Def.__ (Lusztig 2005) Set $X_{\mathbb{T}_r \subset \mathbb{G}_r}$

$$\overline{G}_r \circlearrowright \{ g \in \mathbb{G}_r : g^{-1}\sigma(g) \in \mathbb{U}_r \} \circlearrowleft \overline{T}_r$$

Pos. depth DL ind:

$$R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta) := H_c^*(X_{\mathbb{T}_r \subset \mathbb{G}_r})_\theta .$$

$$\overline{T}_r \to \mathbb{C}^\times$$

r=0: DL theory on the nose.

What DL thms hold for $r>0$?

- $\overline{G}_r$ is a quot of $G_{x,0}$ only if F has char p.

  Stasinski: "mixed char jet sch."

- only some quotients of $G_{x,0}$ arise as $\overline{G}_r$.

  C-Ivanov: framework that starts w/ building.

Conj. (scalar prod. formula).

$\mathbb{T}^1, \mathbb{T}^2$ σ-stable max'l tori in G

θ^1, θ^2 chars of $\overline{T}^1_r, \overline{T}^2_r$.

Then

$$\langle R^{G_r}_{\mathbb{T}^1_r}(\theta^1), R^{G_r}_{\mathbb{T}^2_r}(\theta^2)\rangle_{\overline{G}_r} = \sum_{\substack{w \in W_{\overline{G}_r}(\mathbb{T}^1_r, \mathbb{T}^2_r) \\ = W_{\overline{G}}(\mathbb{T}^1, \mathbb{T}^2)}} \langle \theta^1, {}^{w}\theta^2 \rangle_{\overline{T}_r}.$$

↯ conj. not true in gen.

- 0-toral: Lusztig, Stasinski, C-Ivanov
- general θ. GL_n: C-Ivanov.
- Coxeter wrt $\mathbb{B}$: Dudas-Ivanov.
  Ivanov-Tan-Nie.
- general elliptic $\mathbb{T}'$, θ' <u>Howe factorizable</u>:
  (all θ if p large)
  Chan.

<u>Cor.</u> $\mathbb{T}'$ is elliptic, θ' is regular
then $R_{\mathbb{T}'_r}^{\mathbb{G}_r}(\theta')$ is irreducibe.

<u>Thm.</u> (C-Oi) $s, u \in \overline{G}_r$, $su = us$,

s = p' order

u = p-power order.

$$\text{Ⓛ}_{R^{\mathbb{G}_r}_{\Pi_r}(\theta)}(su) = \frac{1}{|\overline{Z}^0_{\mathbb{G}_r}(s)|} \sum_{g \in \overline{G}_r} \theta^g(s) \cdot \text{Ⓛ}_{Q^{Z^0_{g \cdot r}(s)}_{\Pi_r}(\theta|_{T_{0+}})}(u)$$

$\underbrace{\qquad\qquad}_{\text{pos. depth Green fn.}}$

End of lecture 1:

- $$\mathrm{cInd}_{Z(F)G_{x,0}}^{G(F)}\left(R_{\mathbb{T}}^{\mathbb{G}}(\theta)\right).$$

depth 0 s.c.

Q: $$\mathrm{cInd}_{Z(F)G_{x,0}}^{G(F)}\left(R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)\right) \quad ?$$

- C-Ivanov: $G = GL_n$

- Chen-Stasinski: θ 0-toral ~~yes f. θ~~
- Nie: θ gen.
- Ivanov-Nie-Tan: T coxeter

<u>geom.</u>

- C-Oi:
- 0-toral
- reg. s.c.
- $q \gg 0$

<u>analytic.</u>