ABELIAN VARIETIES OVER FINITE FIELDS: PROBLEM SET 5

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Instructions: The goal of this problem set is to venture into the world of p-divisible groups and Dieudonné modules. Problems marked $(\star)$, $(\star\star)$, and $(\star\star\star)$ denote beginner, intermediate, and advanced problems, respectively.

Notation: As customary, p will be a prime, and q will be a power of p.

In the first two problems, we explore the Newton polygon of a polynomial and use it to define the q-Newton polygon of an abelian variety. These problems are inspired by problems from [Poo06], which serves as a good complementary reference.

Problem 1 $(\star)$
Let K be a field with a non-archimedean valuation $v\colon K^\times \to \mathbb{R}$. The Newton polygon of a polynomial $P(T) = a_0T^n + a_1T^{n-1} + \cdots + a_{n-1}T + a_n$ is the lower convex hull of the finite set $\{(j, v(a_j)) \in \mathbb{R}^2 : 0 \leq j \leq n \text{ and } a_j \neq 0\}$. We will denote it by $\mathcal{N}(P) = \mathcal{N}(P, v)$. We define the width of a line segment from (a, b) to (c, d) (with $a < c$) to be $c - a$.

Theorem A. *Suppose that (K, v) above is complete, so that there is a unique extension v_L of v to any algebraic field extension $L \supset K$. Let $\bar{K}$ be an algebraic closure of K, and let $\bar{v}$ denote the extension of v to $\bar{K}$. Then,*

$$\#\{\alpha \in \bar{K} : P(\alpha) = 0 \text{ and } \bar{v}(\alpha) = s\} = \text{ width of the segment of slope } s \text{ in } \mathcal{N}(P).$$

(1) Prove Theorem A.[a]
(2) Let m be a positive integer. How does $\mathcal{N}(P)$ compare to $\mathcal{N}(P^m)$?
(3) How does the Newton polygon of a product of polynomials relate to the Newton polygons of the factors?

[a] Hint: By changing $P(T)$ to $P(\lambda T)$ for some suitable $\lambda \in \bar{K}$, reduce to the case of slope $s = 0$. Start with $P(T)$ in factored form, and in terms of the number of zeros with positive and negative valuation, determine the location of the slope-zero part of the Newton polygon.

In the context of abelian varieties over finite fields, we focus on the case where $K = \mathbb{Q}_p$, and p is the characteristic of our base field $\mathbb{F}_q$.

Problem 2 $(\star)$
Let the q-valuation $\bar{v}\colon \overline{\mathbb{Q}}_p^\times \to \mathbb{R}$ to be the p-adic valuation renormalized so that $\bar{v}(q) = 1$. We can define the q-Newton polygon of an abelian variety $A/\mathbb{F}_q$ to be the Newton polygon of the characteristic polynomial of Frobenius $P_A(T)$ with respect to the q-valuation $\bar{v}$. We write $\mathcal{N}(A) := \mathcal{N}(P_A(T), \bar{v})$. Newton polygons of g-dimensional abelian varieties over $\mathbb{F}_q$ satisfy the following properties:[a]

a. The left endpoint is $(0, 0)$ and the right endpoint is $(2g, g)$.
b. The vertices are all integer points with nonnegative second coordinate.
c. The vertices are symmetric: If λ is a slope occuring in multiplicity r, then $(1 - \lambda)$ is a slope occuring in multiplicity r. For example, the polygon with vertex $(0, 0), (g, 0), (2g, g)$ is symmetric, since 0 occurs as a slope for g many times and 1 occurs as a slope for g many times.

We say a Newton polygon is admissible if it satisfies properties a, b, c.

(1) Describe the admissible Newton polygons for $g \leq 3$.
(2) Are all admissible Newton polygons realized by some abelian variety of dimension $g \leq 3$? Find explicit examples in the `LMFDB` for each one.
(3) How does the Newton polygon of an abelian variety relate to the Newton polygons of its simple factors in the isogeny category?
(4) How does the q-Newton polygon of A compare to the q^r-Newton polygon of $A_{\mathbb{F}_{q^r}}$?
(5) Calculate the Newton polygon of the varieties described in PSET 4 Problem 11.

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[a]See how many of these you can prove!

The following problem establishes the basics of the ring of Witt vectors attached to a commutative ring. It is taken from [Neu13, Chapter II. Exercise 2-5].

Problem 3 ($\star\star$)

Let $X_0, X_1, \dots$ be an infinite sequence of variables, and p a prime number. For each $n \in \mathbb{Z}_{\geq 1}$, let $W_n(X_0, \dots, X_n) := X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n$.

(1) Show that there exists polynomials $S_0, S_1, \dots ; P_0, P_1, \dots \in \mathbb{Z}[X_0, X_1, \dots ; Y_0, Y_1, \dots]$ such that

$$W_n(S_0, S_1, \dots, S_n) = W_n(X_0, X_1, \dots, X_n) + W_n(Y_0, Y_1, \dots, Y_n)$$
$$W_n(P_0, P_1, \dots, P_n) = W_n(X_0, X_1, \dots, X_n) \cdot W_n(Y_0, Y_1, \dots, Y_n)$$

Now, let A be a commutative ring such that $pA = 0$. Let $\underline{a} := (a_0, a_1, \dots)$ be an infinite tuple with $a_i \in A$. We make the set of such tuples into a commutative ring $W(A)$ as follows. For two such tuples $\underline{a} = (a_0, a_1, \dots), \underline{b} = (b_0, b_1, \dots)$, define addition and multiplication

$$\underline{a} + \underline{b} := (S_0(a, b), S_1(a, b), \dots) \text{ and } \underline{a} \cdot \underline{b} := (P_0(a, b), P_1(a, b), \dots).$$

$W(A)$ is the ring of (p-typical) Witt vectors attached to A.

(2) Check that $1 := (1, 0, \dots)$ is the multiplicative identity of $W(A)$, and that $p := 1 + 1 + \cdots + 1$ is the element $(0, 1, 0, \dots)$ in $W(A)$.
(3) For every Witt vector $\underline{a} = (a_0, a_1, \dots) \in W(A)$, we define the ghost components $a^{(n)}$ as

$$\underline{a}^{(n)} := W_n(\underline{a}) = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n.$$

Consider mappings $V, F : W(A) \to W(A)$ defined by

$$V(\underline{a}) := (0, a_0, a_1, \dots) \text{ and } F(\underline{a}) := (a_0^p, a_1^p, \dots).$$

Show that

$$V(\underline{a})^{(n)} = p\underline{a}^{(n-1)} \text{ and } \underline{a}^{(n)} = (F(\underline{a}))^{(n-1)} + p^n a_n.$$

(4) Now let K be a field of characteristic p. Show that V is a homomorphism of $W(K)$ as an additive group, F is a homomorphism of $W(K)$ as a ring, and

$$V \circ F(\underline{a}) = F \circ V(\underline{a}) = p \cdot \underline{a} = (0, a_0^p, a_1^p, \dots)^a$$

(5) ($\star\star\star$) If K is a perfect field of characteristic p, then $W(K)$ is a complete discrete valuation ring with residue field K and maximal ideal $pW(K)$.
(6) ($\star\star\star$) Show that $W(\mathbb{F}_{p^n}) \cong \mathbb{Z}_{p^n}$, which is the valuation ring of $\mathbb{Q}_{p^n}$, the unique degree n unramified extension of $\mathbb{Q}_p$.

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[a]To show that f, g are the same map from $W(A) \to W(A)$, it suffices to show that $W_n \circ f = W_n \circ g$ from $\mathbb{Z}[\underline{X}; \underline{Y}] \to \mathbb{Z}[\underline{X}; \underline{Y}]$. Also, if A has characteristic p, it suffices to show that $f_n \equiv g_n \pmod{p}$ as an element in $\mathbb{Z}/p\mathbb{Z}[\underline{X}, \underline{Y}]$

The next problem is Exercise 7.4.5 in [BC09], which gives a different way to understand the Witt vectors.

Problem 4 ($\star\star$)
Let k be an arbitrary field of characteristic $p > 0$.

(1) Use the addition law on the truncated Witt ring W_n defined in Problem 3 (applied to all k-algebras), to explain how this gives $\mathbb{A}_k^n$ the structure of a smooth group variety W_n.
(2) Describe the group variety structure explicitly for $n = 2$ and any k.
(3) Recall from PSET 1, Problem 8 the idea of a ring variety. Write down the axioms to define a "commutative ring scheme" and exhibit W_n as such an example.

The following is Lemma/Exercise after Definition 4.28 in [CO09]. It introduces the notion of the Dieudonné ring and the local Cartier ring.

Problem 5 ($\star$)
Let K be a perfect field of characteristic p. Let $W(K)$ be the ring of Witt vectors and let $\sigma\colon W(K) \to W(K)$ be the homomorphism $(a_0, a_1, \ldots) \mapsto (a_0^p, a_1^p, \ldots)$. The Dieudonné ring D_K is defined to the polynomial ring $W(K)[F, V]$ satisfying $FV = VF = p$, $F\underline{a} = \underline{a}^\sigma F$, $V\underline{a}^\sigma = \underline{a}V$.

(1) Show that the Dieudonné ring D_K can be naturally identified with the $\mathbb{Z}$-graded ring $\bigoplus_{i\in\mathbb{Z}} c_i V^i W(K)$ with the relation $\underline{a}V^n = V^n \underline{a}^{\sigma^n}$, where $c_i = p^{-i}$ if $i < 0$ and $c_i = 1$ otherwise. This means $W(k)[F, V]$ is the ring consisting of finite sums $\sum_i a_i V^i$ where $a_i \in W(K)$, $v_K(a_i) \geq \max\{0, -i\}$.
(2) Let $W(K)[[V, F\rangle\rangle$ be the ring consisting of formal Laurent series $\sum_i a_i V^i$ where $a_i \in W(K)$, $v_K(a_i) \geq \max\{0, -i\}$, and $v_p(a_i) + i \to \infty$ as $|i| \to \infty$. Again the relation $\underline{a}V^n = V^n \underline{a}^{\sigma^n}$ is given. Let $v\colon W(K)[[V, F\rangle\rangle \to \mathbb{Z}$ be defined by $v(\sum_i a_i V^i) = \min_i\{v_K(a_i) + i\}$. Show that v is a discrete valuation on $W(K)[[V, F\rangle\rangle$. [a]
(3) Show that the inclusion $W(K)[F, V] \hookrightarrow W(K)[[V, F\rangle\rangle$ is a ring homomorphism whose image is dense. [b]

[a] This ring can be naturally identified with the local Cartier ring $\mathrm{Cart}_p(K)$.
[b] This indicates that the Dieudonné ring can be naturally identified as a dense subring of the local Cartier ring.

We compute the Cartier duals of some finite flat group schemes.

Problem 6 ($\star\star$)
Let k be a field. Compute the Cartier duals of the following commutative k-groups.

(1) $\underline{\mathbb{Z}/n\mathbb{Z}}$. Recall that as a k-scheme, this is given by $\operatorname{Spec} A$ where $A := \prod_{i\in\mathbb{Z}/n\mathbb{Z}} e_i k$. The multiplication on A is defined by $e_i \cdot e_j = \delta_{ij} e_i$, and the co-multiplication is given by $\Delta(e_r) = \sum_{i+j=r} e_i \otimes e_j$.
(2) When k has characteristic p, the group $\alpha_p := \operatorname{Spec} k[x]/(x^p)$, considered as a subgroup of $\mathbb{G}_{a,k}$.

In problem 7 and 8, we use Dieudonné modules to classify the commutative finite flat group schemes of order p defined over an algebraically closed field k of characterstic p, and apply this to study the p-torsion group scheme of a supersingular elliptic curve over k. If you get stuck, the solutions can be found here.

Problem 7 ($\star$)
Let k be an algebraically closed field of characterstic p. Let $D_k = W(k)[F, V]$ be the Dieudonné ring.

(1) Using [BC09, Theorem 7.2.4], there is an equivalence of categories between commutative order p finite flat group schemes over k and left D_k-modules M whose underlying $W(k)$-module is of length 1. Use (6) from Problem 3 to show that such an M must be isomorphic to $W(k)/(p)$ as a $W(k)$-module.
(2) To specify the D_k-module structure on M, it suffices to write down the action of F and V. Let e be a basis element of M as a 1-dimensional k-vector space. Let $\alpha, \beta \in k$ be such that
$$Fe = \alpha e, \quad Ve = \beta e.$$
Show that at least one of α, β is zero.

(3) Conversely, show that upon fixing a basis element e, any choice of (α, β) with at least one of α and β being 0 uniquely determines a Dieudonné module over $W(k)$ of length 1.

(4) Show that upon changing the basis $e' := \lambda e$ for some $\lambda \in k^\times$, then if one of α, β is nonzero, it can be chosen to be 1.

(5) Now we have reduced to the cases (α, β) being $(0,0)$, $(1,0)$, or $(0,1)$. There are three well-known finite flat group schemes of order p over a characteristic p field: μ_p, $\underline{\mathbb{Z}/p\mathbb{Z}}$, and α_p. For each group scheme, find out whether it is connected, étale, or neither.

(6) Show that the relative Frobenius kills a connected order p group scheme over k, and is an isomorphism on an étale group scheme.[a] Deduce that the $(1,0)$ Dieudonné module must correspond to $\underline{\mathbb{Z}/p\mathbb{Z}}$.

(7) Use the definition of the Verschiebung morphism on a group scheme together with Problem 6 to decide which of μ_p, α_p correspond to $(0,1)$, and which to $(0,0)$.

[a] Hint: See these notes by Andrew Snowden.

Problem 8 ($\star\star$)

Let $E/\overline{\mathbb{F}}_p$ be a supersingular elliptic curve. We will show there is a unique group scheme G over $\overline{\mathbb{F}}_p$ of order p^2 such that $E[p] \cong G$.

(1) Using [BC09, Theorem 7.2.4] again, a group scheme G over k of order p^2 corresponds to a Dieudonné module $M(G)$ of length 2 as a $W(\overline{\mathbb{F}}_p)$-module. Show that if G is p-torsion, then so is $M(G)$. In particular, $M(G)$ must be isomorphic to $W(\overline{\mathbb{F}}_p)/(p) \oplus W(\overline{\mathbb{F}}_p)/(p)$ as a $W(\overline{\mathbb{F}}_p)$-module.

(2) ($\star\star$) Use the connected-étale sequence and the fact that $\#E[p](\overline{\mathbb{F}}_p) = 1$ to show that $E[p]$ is connected.

(3) ($\star\star$) As an extension of Part (6) of Problem 7, one can show the relative Frobenius ϕ_G is a finite flat morphism of degree p, and is nilpotent on any connected finite flat group scheme G over a field. Use this to show that the kernel of ϕ_G is an order p flat group scheme, and so the Dieudonné module of $\ker(\phi_G)$ must be isomorphic to $\overline{\mathbb{F}}_p$ as a $W(\overline{\mathbb{F}}_p)$-module.

(4) The induced action of Frobenius on the Dieudonné module $M(E[p])$ is also nilpotent by functoriality, so we can choose an $\overline{\mathbb{F}}_p$-basis e_1, e_2 of $M(E[p])$ so that

$$Fe_1 = e_2, \quad Fe_2 = 0.$$

Show that $Ve_2 = 0$, and $Ve_1 = \alpha e_2$ for some $\alpha \in \overline{\mathbb{F}}_p$. Show that $\alpha \neq 0$.

(5) By scaling e_1 and using that $\overline{\mathbb{F}}_p$ is algebraically closed, show that we can let $\alpha = 1$. In particular, there is a unique Dieudonné module corresponding to the group scheme $E[p]$ for a supersingular elliptic curve.

The case of E ordinary is more straightforward. Use the fact that $\#E[p](\overline{\mathbb{F}}_p) = p$ and the fact that the connected-étale exact sequence splits for group schemes over a perfect field to show that $E[p] \cong \mu_p \times \underline{\mathbb{Z}/p\mathbb{Z}}$.

The following problem is adapted from [CO09, Exercise 4.6]. Here we investigate the endomorphism algebra of simple Dieudonné modules over an algebraically closed base field.

Problem 9 ($\star\star$)

Let k be an algebraically closed field containing $\mathbb{F}_p$. Let D_k be the Dieudonné ring as in Problem 5, and $D_k[\frac{1}{p}]$ be the rational Dieudonné ring. Now, let (m, n) be a pair of non-negative integers such that $\gcd(m, n) = 1$. Let $N_{m,n} := D_k[\frac{1}{p}]/D_k[\frac{1}{p}](F^m - V^n)$. $N_{m,n}$ is a simple object in the isogeny category of Dieudonné module over k. We want to compute $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$.

(1) Show that $N_{m,n} \cong D_k[\frac{1}{p}]/D_k[\frac{1}{p}](F^{m+n} - p^n)$.

(2) Let $\varphi \in \mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$. Suppose $\varphi(1) = \sum_{i=0}^{m+n-1} a_i F^i$ with $a_i \in W(k)[\frac{1}{p}]$. Use the fact that $(F^{m+n} - p^n)\varphi(1) \in D_k[\frac{1}{p}](F^{m+n} - p^n)$ to show that all the a_i's lie in $W(\mathbb{F}_{p^{m+n}})[\frac{1}{p}] = \mathbb{Q}_{p^{m+n}}$[a]

(3) Show that the center of $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$ is $\mathbb{Q}_p$.

(4) Use the fact that $N_{m,n}$ is a simple left $D_k[\frac{1}{p}]$-module, show that $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$ is a central division algebra over $\mathbb{Q}_p$.

(5) Recall the definition and notation of $D_{p,h,n}$ from PSET 2, Problem 4. It can be written as $\mathbb{Q}_{p^h}[F]/(F^h - p^n)$, where $F\alpha = \alpha^\sigma F$ for $\alpha \in \mathbb{Q}_{p^h}$. Show that $\varphi \mapsto \varphi(1)$ gives an isomorphism $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n}) \cong \mathbb{Q}_{p^{m+n}}[F]/(F^{m+n} - p^n)$.

(6) Conclude that $\mathrm{End}_{D_k[\frac{1}{p}]}(N_{m,n})$ is a central simple algebra over $\mathbb{Q}_p$ with Hasse-invariant $\frac{n}{m+n}$.

[a] That is, show that $a_i^{\sigma^{m+n}} = a_i$ for all a_i.

The next problem is Exercise 7.4.8 in [BC09]. It displays the role p-divisible groups play compared to ℓ-adic Tate-modules: they are more suitable for encoding information at p!

Problem 10 ($\star\star\star$)
Let A and B be abelian varieties over a perfect field k of characteristic $p > 0$. Recall that there is an additive antiequivalence of categories $G \mapsto \mathbb{D}(G)$ between the category of p-divisible groups over k and the category of left $W(k)[F, V]$-modules which are also finite as $W(k)$-modules.

(1) Show that the natural map
$$\mathrm{Hom}_k(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{Hom}_{W(k)[F,V]}(\mathbb{D}(B[p^\infty]), \mathbb{D}(A[p^\infty]))$$
is injective.

(2) Show, however, that the natural map
$$\mathrm{Hom}_k(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{Hom}_{\mathbb{Z}_p}(T_pA, T_pB)$$
is never injective[a].

(3) Now require k to be finite. If $f \in \mathrm{End}_k(A)$ is a nonzero endomorphism of A then the common characteristic polynomial $P_f \in \mathbb{Z}[T]$ of all $T_\ell(f) \in \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell A)$ with $\ell \neq \mathrm{char} k$ is also the characteristic polynomial of $\mathbb{D}(f) \in \mathrm{End}_{W(k)}(\mathbb{D}(A[p^\infty]))$.

[a] Here $T_p(A) = TA[p^\infty](\bar{k})$ (see Problem 14) is the "naive" Tate module

In Problem 7, we have considered examples of finite flat group schemes of order p. The following problem expands on these examples to give examples of p-divisible groups of height 1.

Problem 11 ($\star\star$)
Let k be an algebraically closed field of characteristic p.

(1) Let $\mathbb{G}_m/k$ be the multiplicative group scheme defined over k.
  (a) Show that the multiplication $[p^i]$ is given by $x \mapsto x^{p^i}$ on the coordinate ring. Determine the Hopf algebra of the group scheme $\mathbb{G}_m[p^i]$, i.e. the kernel of $[p^i]$.
  (b) Define $G_i := \mathbb{G}_m[p^i]$. Show that $\mathbb{G}_m[p^\infty] := \{G_i\}_{i\geq 1}$, together with the inclusion $j_i : G_i \to G_{i+1}$, is a p-divisible group of height 1. This p-divisible group is often denoted μ_{p^∞}.
  (c) Show that the relative Frobenius $\mathrm{F}_{G_i/k} : G_i \to G_i^{(p)} \cong G_i$, agrees with $[p] : G_i \to G_i$. Conclude that $V_{G_i/k} : G_i \to G_i$ is the identity.
  (d) Let $G_{m,n}$ be the p-divisible group whose Dieudonné module is $M_{m,n} := D_k/D_k(F^m - V^n)$. By comparing the action of Frobenius and Verschiebung and using the Dieudonné-Mannin classfication[a], show that μ_{p^∞} is isogenous to $G_{0,1}$. That is, $\mathbb{D}(\mu_{p^\infty}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong M_{0,1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

(2) Let $H_i = \underline{p^{-i}\mathbb{Z}/\mathbb{Z}}_k$ be the constant group scheme over k attached to the finite group $p^{-i}\mathbb{Z}/\mathbb{Z}$.
  (a) Show that $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_k := \{H_i\}_{i\geq 1}$, together with the inclusion $j_i : H_i \to H_{i+1}$, is a p-divisible group of height 1.
  (b) Show that $\mathrm{F}_{H_i/k} : H_i \to H_i^{(p)} \cong H_i$ is the identity. Conclude that $V_{H_i/k}$ is $[p]$.
  (c) Show that $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_k$ is isogenous to $G_{1,0}$.[b]

[a]Use the statement of [BC09, Theorem 8.1.4]. $M_{m,n} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is $D_{m,m+n}$ in the notation of Theorem 8.1.4, and is $N_{m,n}$ in the notation of Problem 9.

[b]Use the fact that $\mathbb{D}(G^t) = \mathbb{D}(G)^\vee$ and $M_{m,n}^\vee = M_{n,m}$, we see that $\mathbb{G}_m[p^\infty]$ is the Serre dual (see below) of $\underline{\mathbb{Q}_p/\mathbb{Z}_p}_k$.

The following problem gives the construction of Dieudonné module associated to the Serre dual of a p-divisible group.

Problem 12 ($\star\star$)

Let k be an algebraically closed field of characteristic p. Let M be a Dieudonné module[a], i.e. a finite free $W(k)$-module with left D_k action. We construct its dual $M^\vee$ as follows. As a $W(k)$ module, $M^\vee = \mathrm{Hom}_{W(k)}(M, W(k))$, with the action of V and F given as

$$(V \cdot h)(m) = (h(F(m)))^{\sigma^{-1}}, \ (F \cdot h)(m) = (h(V(m)))^{\sigma}$$

for all $h \in M^\vee$ and $m \in M$.

(1) For a pair of non-negative integers (m, n) such that $\gcd(m, n) = 1$, let $M_{m,n}$ be as in Problem 11. Show that $M_{m,n}^\vee \cong M_{n,m}$.

(2) We have the following facts:
- Let X/k an abelian variety. Let $X[p^\infty]$ denote its p-divisible group, and $X[p^\infty]^{\mathrm{t}}$ the Serre dual of $X[p^\infty]$, then
$$X[p^\infty]^{\mathrm{t}} \cong X^\vee[p^\infty],$$
where $X^\vee$ is the dual abelian variety.
- If G is a p-divisible group over k, and $D(G)$ is its Dieudonné module, then
$$D(G^t) \cong D(G)^\vee$$

Use the above facts, show that the Newton polygon of an abelian variety is symmetric. That is, $X[p^\infty]$ is isogenous to $\bigoplus_i (G_{m_i,n_i} \oplus G_{n_i,m_i})^{r_i}$ for some (m_i, n_i) non-negative and $\gcd(m_i, n_i) = 1$.

[a]There is an unfortunate clash of terminology with the Dieudonné module of a finite flat group scheme, which isn't necessarily torsion-free. We hope that the meanings are clear from the context.

The following problem explores examples of p-divisible groups attached to an abelian variety.

Problem 13 ($\star\star$)

(1) Recall that if $f : X \to Y$ is an isogeny between abelian varieties over a field k, then $\deg(f) = \mathrm{rank}(\ker(f))$, i.e. the rank of the finite group scheme $\ker(f)$ over k. Show that the p-divisible group of a g-dimensional abelian variety over k is of height $2g$.

(2) Now let $E/\mathbb{F}_q$ be an elliptic curve.
  (a) Suppose $E/\mathbb{F}_q$ is supersingular. Recall in PSET 3, problem 7, we have shown that $\mathrm{End}^0(E) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong D_{p,2,1}$, the central division algebra over $\mathbb{Q}_p$ with Hasse-invariant $\frac{1}{2}$. Combine Problem 10 and Problem 9 Part (5) to conclude that $E_{\overline{\mathbb{F}}_q}[p^\infty]$ is isogenous to $G_{1,1}$.
  (b) Suppose $E/\mathbb{F}_q$ is ordinary. Recall in PSET 3, problem 9, we have shown that $L = \mathrm{End}^0(E)$ is an imaginary quadratic extension over $\mathbb{Q}$ generated by ϕ_q. Furthermore, the characteristic polynomial of ϕ_q is $T^2 - aT + q$, where $v_p(a) = 0$. Show that $L \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$. Use the injection
$$\mathrm{End}^0(E) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{End}^0(E_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \mathrm{End}(E_{\overline{\mathbb{F}}_p}[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$
to conclude that $E_{\overline{\mathbb{F}}_q}[p^\infty]$ is isogenous to $G_{1,0} \oplus G_{0,1}$.

(3) Recall that in PSET 4, Problem 11, for a pair of non-negative integers (m, n) with $n < m$ and $\gcd(m, n) = 1$, we have a simple abelian variety $A/\mathbb{F}_q$ of dimension $g = m + n$, and the Frobenius ϕ_q on A has minimal polynomial $h_A(T) = T^2 - p^n T + p^g$. Moreover, $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong D_{p,g,m} \oplus D_{p,g,n}$. Use these to show that $A_{\overline{\mathbb{F}}_q}[p^\infty]$ is isogenous to $G_{n,m} \oplus G_{m,n}$.

As an important notion to study p-divisible groups, we introduce the Tate module of a p-divisible group.

Problem 14 ($\star\star$)
Let G be a p-divisible group over an affine perfect scheme S of characteristic p. Consider the inverse limit

$$TG := \varprojlim_{\times p} G[p^n].$$

Show that this limit exists in the category of schemes and TG is an scheme, flat over S. This is called the (schematic) Tate module of the p-divisible group G.[a]

(1) Show that the functor of points of TG identifies with the following functor

$$(T \to S) \mapsto \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G_T),$$

where $\mathbb{Q}_p/\mathbb{Z}_p$ is the constant p-divisible group over T, and G_T denotes the base change.

(2) Show that over a quasicompact noetherian test scheme U of characteristic p, the Tate module $T\mu_{p^\infty}(U)$ is trivial.

[a]Depending on conventions, sometimes the Tate module of G refers to the set of $\bar{k}$-points of TG, which is a finite free $\mathbb{Z}_p$-module.

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