3. LECTURE 3: LOWER BOUNDS FOR CANONICAL HEIGHTS

Today we’ll be moving from $K$ to $\bar{K}$, so we need to normalize our heights appropriately. This can be done (see notes) and gives height functions

$$h : \bar{K} \rightarrow [0, \infty),$$
$$\hat{h}_D : A(\bar{K}) \rightarrow [0, \infty).$$

[We always take $D \in \text{Div}(A)$ ample and symmetric.]

$\hat{h}_D$ is “canonical” for $A(\bar{K})$. Similarly $h$ is “canonical” for the group $\mathbb{G}_m(\bar{K}) = \bar{K}^\ast$.

$$\hat{h}_D : A(K) \rightarrow [0, \infty) \text{ satisfies } \hat{h}_D[m]P = m^2\hat{h}_D(P).$$

$$h : \mathbb{G}_m(K) \rightarrow [0, \infty) \text{ satisfies } h(\alpha^m) = |m| \cdot h(\alpha).$$

Where do they vanish?

\begin{align*}
\hat{h}_D(P) = 0 & \iff P \in A(\bar{K})_{\text{tors}} \\
\text{versus} & \\
h(\alpha) = 0 & \iff \alpha \in \mathbb{G}_m(\bar{K})_{\text{tors}} = \{\text{roots of unity}\}.
\end{align*}

This raises a fundamental question:

How small can the canonical height be, if it’s not zero?

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**Intuition:** Small non-zero canonical height requires a large field extension.

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**Example 1:** For $\mathbb{G}_m(\overline{\mathbb{Q}})$,

$$h(2^{1/n}) = \frac{1}{n} h(2), \quad \text{but} \quad [\mathbb{Q}(2^{1/n}) : \mathbb{Q}] = \#\mu_n = n.$$  

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**Example 2:** Similarly, for $P \in A(K)$, we can take $Q \in [n]^{-1}(P)$ to get

$$\hat{h}_D(Q) = \frac{1}{n^2} \hat{h}_D(P), \quad \text{but} \quad [K(Q) : K] \approx \#A[n] = n^{2g}.$$  

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**Classical** Lehmer Conjecture: There is a constant $C > 0$ such that

$$h(\alpha) \geq \frac{C}{[\mathbb{Q}(\alpha) : \mathbb{Q}]} \quad \text{for all } \alpha \in \mathbb{G}_m(\overline{\mathbb{Q}}) \setminus \mathbb{G}_m(\overline{\mathbb{Q}})_{\text{tors}}.$$  

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Best proven result: **Theorem** (Dobrowolski, 1979):

$$h(\alpha) \geq \frac{C'}{d} \left( \frac{\log \log d}{\log d} \right)^3 \quad \text{for all } \alpha \in \mathbb{G}_m(\overline{\mathbb{Q}}) \setminus \mathbb{G}_m(\overline{\mathbb{Q}})_{\text{tors}}$$

where $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$.  

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Lehmer Conjecture for Abelian Varieties:
(Masser 1984): Let \( g = \dim(A) \). There is a constant \( C(A/K, D) > 0 \) such that

\[
\hat{h}_D(P) \geq \frac{C(A/K)}{[K(P) : K]^{1/g}} \quad \text{for all } P \in A(\bar{K}) \setminus A(\bar{K})_{\text{tors}}.
\]

Some partial results (due to a bunch of different people, see notes):

**Theorem:** Let

\[
d = d(K, P) = [K(P) : K], \quad C = C(A/K, D, \epsilon).
\]

Then

(a) \( \hat{h}_D(P) \geq C/d^{2g+1+\epsilon} \) for all \( A/K \).
(b) \( \hat{h}_D(P) \geq C/d^{1+\epsilon} \) if \( A/K \) has CM.
(c) \( \hat{h}_D(P) \geq C/d^{2+\epsilon} \) if \( g = 1 \) and \( j(A) \notin R_K \).

There are two methods that have been used for Lehmer’s problem (called informally):

1. **Transcendence Theory Method**
2. **Fourier Averaging Method**

**Transcendence Theory Method:** Let \( L/K \) with \( d = [L : K] \). Look at

\[
A(L, B) := \{ P \in A(L) : \hat{h}_{A,D}(P) \leq B \}.
\]

Goal is to show that

\[
\#A \left( L, \frac{C_1}{d} \right) \leq C_2 d^{g+\epsilon}.
\]
Exploit group law by considering
\[ A(L, B)^{(g)} := \{ P_1 + \cdots + P_g : P_1, \ldots, P_g \in A(L, B) \}, \]
so
\[ \#A(L, B)^{(g)} \approx \frac{\#A(L, B)^g}{g!} \quad \text{lots of points}, \]
\[ A(L, B)^{(g)} \subseteq A(L, g^2 B) \quad \text{with height not too large.} \]

Then

1. Construct a non-zero “small” (theta) function \( F \) on \( A \) that vanishes to high order at the points in \( A(L, B)^{(g)} \).
2. Use Cauchy’s theorem to get upper bound for partial derivatives \( |\partial F(Q)| \) for \( Q \in A(L, B)^{(g)} \).
3. Use a zero-estimate from transcendence theory to get a lower bound for partial derivatives \( |\partial F(Q)| \) for \( Q \in A(L, B)^{(g)} \).
4. If \( \#A(L, B) \) is sufficiently large, the upper and lower bounds contradict.

**Note:** Many details have been omitted!!

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**Fourier Averaging Method:**
Up to now, primarily applied to \( \dim(A) = 1 \). [If time at end, say a few words about this.]
3.2. **Small Heights for a Fixed $K$ and Varying $A/K$.** Returning to our fundamental question:

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How small can the canonical height be, if it’s not zero?
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We move in an orthogonal direction:

**Intuition:** Small non-zero canonical height of a point requires a “complicated” abelian variety.

First step: How to measure the “complexity” (height) of an abelian variety. Some possibilities:

- Define $h(A/K)$ to be the smallest height of the coefficients of polynomials that describe a projective embedding of $A/K$. E.g. For an elliptic curve

  $$E : y^2 = x^3 + Ax + B,$$

  set

  $$h(E/K) := \min_{u \in K^*} h([1, u^4 A, u^6 B]).$$

- Let $\mathcal{A}_g \subset \mathbb{P}^N$ be the moduli space of principally polarized abelian varieties of dimension $g$ with a projective embedding. Then define

  $$h(A/K) := h_{\mathcal{A}_g}(j(A)) + \log N_{K/Q}(\text{Conductor}(A/K)).$$

  Here $j(A) \in \mathcal{A}_g(K)$ is the moduli point associated to $A$.

**Note:** We need the conductor in (2) to deal with twists, i.e., abelian varieties with the same $j(A)$.
Dem’janenko–Lang Height Conjecture:
Let $P \in A(K)$ satisfy $\mathbb{Z} \cdot P$ is Zariski dense in $A$. Then
\[
\hat{h}_D(P) \geq C_1(K, g) \cdot h(A/K) - C_2(K, g).
\]

Some Partial Results (by various people):
(I) The DLH conjecture is true if $j(A)$ is at least $\varepsilon$-distance away from the boundary of $A_g(\mathbb{C})^{\text{simple}}$.
(II) The DLH conjecture is true for twists, i.e., for a fixed value of $j(A)$.
(III) For $\dim(A) = 1$, the DLH conjecture is true for $A$ with bounded Szpiro ratio $\frac{\log |\text{Disc}|}{\log |\text{Cond}|}$. In particular,

$abc$-conjecture $\implies$ DLH conjecture for elliptic curves.

A brief word about the proofs:
(I) Transcendence theory method.
(II) $K$-rational points on twists give points in fields with large discriminant on original abelian variety.
(III) Fourier averaging method.