2. Lecture 2: Applications & Local Height Functions

Review: For ample symmetric $D \in \mathrm{Div}(A)$,

$$\hat{h}_D : A(K) \longrightarrow [0,\infty), \quad \hat{h}_D(P) = \lim_{n\to\infty} 4^{-n} h_D\bigl([2^n]P\bigr),$$

$$\langle P, Q\rangle_D := \frac{1}{2}\Bigl(\hat{h}_D(P+Q) - \hat{h}_D(P) - \hat{h}_D(Q)\Bigr),$$

$$\|P\|_D := \sqrt{\langle P, Q\rangle_D}.$$

We give $A(K)_{\mathbb{R}}$ the structure of Euclidean vector space via the positive definite bilinear form

$$\left\langle \sum_{i=1}^r a_i P_i,\ \sum_{j=1}^r b_j P_j \right\rangle_D := \sum_{i=1}^r \sum_{j=1}^r a_i b_j \langle P_i, P_j\rangle_D.$$

Note on Notation: Every element of

$$A(K)_{\mathbb{R}} := A(K) \otimes \mathbb{R}$$

can be written as a finite sum $\sum P_i \otimes a_i$, and for notational convenience, we've written this sum as $\sum a_i P_i$.

2.1. **Application: Counting Points.**

Néron invented $\hat{h}_D$ to give an improved count of points in $A(K)$.

Theorem (Néron).

A/K	an abelian variety defined over a number field.
D	an ample symmetric divisor on A.
h_D	an associated Weil height.
r	the rank of $A(K)$.

Consider the counting function

$$\mathcal{N}\bigl(A(K), h_D, T\bigr) := \#\bigl\{P \in A(K) : h_D(P) \le T\bigr\}.$$

There is a constant $\alpha(A/K, D) > 0$ such that

$$\mathcal{N}\bigl(A(K), h_D, T\bigr) = \alpha(A/K, D)T^{r/2} + O(T^{(r-1)/2}) \quad \text{as } T \to \infty.$$

More precisely, we may take

$$\alpha(A/K, D) = \#A(K)_{\text{tors}} \cdot \frac{\operatorname{Vol}\bigl\{\mathbf{x} \in \mathbb{R}^r : \|\mathbf{x}\| \le 1\bigr\}}{\operatorname{Reg}_D(A/K)^{1/2}}.$$

Proof Sketch.
Step 1. Since

$$\hat{h}_D = h_D + O(1),$$

it suffices to count

$$\mathcal{N}\bigl(A(K), \hat{h}_D, T\bigr) := \#\bigl\{P \in A(K) : \hat{h}_D(P) \le T\bigr\}.$$

Step 2. We let (abuse of notation)

$$A(K)_{\mathbb{Z}} := \operatorname{Image}\Bigl(A(K) \longrightarrow A(K)_{\mathbb{R}}\Bigr).$$

$A(K)_{\mathbb{Z}}$ is called the *Mordell–Weil lattice*. This gives an exact sequence

$$0 \longrightarrow A(K)_{\text{tors}} \longrightarrow A(K) \longrightarrow A(K)_{\mathbb{Z}} \longrightarrow 0.$$

This yields

$$\begin{aligned}\mathcal{N}\bigl(A(K),&\hat{h}_D, T\bigr)\\ &= \#A(K)_{\text{tors}} \cdot \mathcal{N}\bigl(A(K)_{\mathbb{Z}}, \hat{h}_D, T\bigr)\\ &= \#A(K)_{\text{tors}} \cdot \mathcal{N}\bigl(A(K)_{\mathbb{Z}}, \|\cdot\|_D^2, T\bigr)\\ &= \#A(K)_{\text{tors}} \cdot \mathcal{N}\bigl(A(K)_{\mathbb{Z}}, \|\cdot\|_D, T^{1/2}\bigr).\end{aligned}$$

Step 3. Use a standard "counting lattice points in a (convex) expanding domain" result, such as:
Theorem.

- V $\mathbb{R}$-vector space with Euclidean norm $\|\cdot\|$.
- L a lattice in V.
- $\mathcal{F}_L$ a fundamental domain for L.
- $\mathcal{B}$ the unit ball $\bigl\{\mathbf{v} \in V : \|\mathbf{v}\| \le 1\bigr\}$.

Then

$$\bigl\{\mathbf{v} \in L : \|\mathbf{v}\| \le T\bigr\} = \frac{\operatorname{Vol}(\mathcal{B})}{\operatorname{Vol}(\mathcal{F}_L)} \cdot T^{\dim V} + O(T^{\dim V - 1}).$$

We apply the theorem with:

$$V = A(K) \otimes \mathbb{R}$$

$$L = A(K)_{\mathbb{Z}} = \operatorname{Image}\bigl(A(K) \longrightarrow A(K) \otimes \mathbb{R}\bigr)$$

$$\|\cdot\| = \|\cdot\|_D \quad \text{(canonical height norm)}$$

$$\operatorname{Vol}(\mathcal{F}_L) = \operatorname{Reg}_D(A/K)^{1/2} \quad \text{(Néron–Tate regulator)}$$

2.2. Local Height Functions. We consider

v	a place of K, i.e., $v \in M_K$.
K_v	completion of K at v.
P	a point in $\mathbb{P}^N(K_v)$.
D	an effective divisor in $\mathrm{Div}(\mathbb{P}^N)$.

We would like to define

$$\lambda_{D,v}(P) := \begin{pmatrix}\text{the } v\text{-adic local height} \\ \text{of } P \text{ with respect to } D\end{pmatrix}$$
$$:= -\log \begin{pmatrix}\text{the } v\text{-adic distance} \\ \text{from } P \text{ to } D \text{ in } \mathbb{P}^N(K_v)\end{pmatrix}.$$

Note: $\lambda_{D,v}(P)$ is big $\iff$ P is v-adically close to D.

Formally, we choose an equation for D, say

$D = \{F_D(\mathbf{x}) = 0\}$, for a homogeneous degree d polynomial $F_D \in K[x_0, \dots, x_N]$,

and we set

$$\lambda_{D,v}(P) := -\log\min\left\{\left|\frac{F(P)}{x_0(P)^d}\right|_v, \cdots, \left|\frac{F(P)}{x_n(P)^d}\right|_v\right\}.$$

Then (for properly normalized absolute values) we have

$$h_{\mathbb{P}^N}(P) = \sum_{v \in M_K} \lambda_{D,v}(P) + O(1) \quad \text{for } P \in \mathbb{P}^N(K) \smallsetminus |D|.$$

More generally one can construct local heights $\lambda_{X,D,v}$ on any (smooth) variety, but they are only well-defined up to bounded functions. For abelian varieties, Néron constructed canonical local heights that are well-defined up to constant functions. He gave three constructions:

- A limit process building on Tate's construction.
- Using intersection theory for non-archimedean v
- Using complex analysis (theta functions) for archimedean v.

Theorem. (Néron) There exists a (unique) collection of functions

$$\hat{\lambda}_{D,v} : A(K_v) \smallsetminus |D| \longrightarrow \mathbb{R},$$

indexed by divisors $D \in \mathrm{Div}_K(A)$ and absolute values $v \in M_K$, so that:

(a) $\hat{\lambda}_{D,v}$ is continuous for the v-adic topology on $A(K_v)$.

(b) $\hat{\lambda}_{D+D',v} = \hat{\lambda}_{D,v} + \hat{\lambda}_{D',v} + \gamma_v$.

(c) $\hat{\lambda}_{A,\varphi^*D,v} = \hat{\lambda}_{B,D,v} \circ \varphi + \gamma_v$ for isogenies $\varphi : A \to B$.

(d) (Normalization)

$$\lim_{N\to\infty} N^{-2g} \sum_{\substack{P\in A[N]\\ P\notin|D|}} \hat{\lambda}_{D,v}(P) = 0.$$

[Really working in extensions L_w/K_v.]

(e) (Local-Global Decomposition)

$$\hat{h}_D(P) = \sum_{v\in M_K} \hat{\lambda}_{D,v}(P) - \underbrace{\kappa(A,D)}_{\text{Vanishes if } \dim(A)=1}.$$

Explicit Formulas for Local Heights:
(as time permits)

Good Reduction: Suppose that A/K has good reduction at $v \in M_K^\circ$. Let

$$\begin{aligned} \mathcal{A}/R_K &= \text{Néron model of } A/K \\ \overline{D} &= \text{closure in } \mathcal{A} \text{ of the divisor } D \in \operatorname{Div}(A/K) \\ \overline{P} &= \text{closure in } \mathcal{A} \text{ of the point } P \in A(K) \end{aligned}$$

Then

$$\hat{\lambda}_{A,D,v}(P) = \underbrace{\left\langle \overline{D} \cdot \overline{P} \right\rangle_{\mathcal{A},v}}_{\text{Intersection index of } \overline{D} \text{ and } \overline{P} \text{ on the fiber over } v.}$$

Bad Reduction: Let

$$j_v : A(K) \longrightarrow (\mathcal{A}/\mathcal{A}^\circ)_v(k_v)$$

be the homomorphism that sends a point to its image in the group of components of the Néron model over v. Then there is a function

$$\mathbb{B}_{D,v} : (\mathcal{A}/\mathcal{A}^\circ)_v(k_v) \longrightarrow \mathbb{R}$$

so that

$$\hat{\lambda}_{A,D,v}(P) = \left\langle \overline{D} \cdot \overline{P} \right\rangle_{\mathcal{A},v} + \mathbb{B}_{D,v}\bigl(j_v(P)\bigr) - \kappa(A,D,v).$$

Here $\kappa(A,D,v)$ is to make normalization hold.

Archimedean Absolute Values: Write

$$A(\bar{K}_v) = A(\mathbb{C}) = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g).$$

There is an associated theta divisor Θ_τ and a theta function $\theta_\tau(\mathbf{z})$. Then

$$\hat{\lambda}_{A_\tau,\Theta_\tau} : A_\tau(\mathbb{C}) \smallsetminus \operatorname{Support}(\Theta_\tau) \longrightarrow \mathbb{R}$$

is given by

$$\begin{aligned}\hat{\lambda}_{A_\tau,\Theta_\tau}(\mathbf{z}) = &-\log\bigl|\theta_\tau(\mathbf{z})\bigr| \\ &+ \pi \cdot {}^t(\operatorname{Im}\mathbf{z}) \cdot (\operatorname{Im}\tau)^{-1} \cdot (\operatorname{Im}\mathbf{z}) + \kappa_\tau.\end{aligned}$$