# 1. Lecture 1: Construction and Properties of Canonical Heights

$K$ is a characteristic 0 field, typically:
- a number field or 1-dim'l function field.
- a complete local field.

## Heights on $\mathbb{P}^N$

Informal (see notes for formal) definition:

$$h : \mathbb{P}^N(K) \longrightarrow [0, \infty),$$
$$h_{\mathbb{P}^N}(P) \approx \#\text{ of bits needed to describe } P.$$

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## Weil Height Machine Overview

- $X/K$, a smooth projective variety.
- $D, D', \ldots,$ divisors (on $X$).

(a) **Normalization**:

$$\begin{pmatrix} D \text{ very ample,} \\ f_D : X \hookrightarrow \mathbb{P}^N \end{pmatrix} \implies h_D(P) = h_{\mathbb{P}^N}\bigl(f_D(P)\bigr) + O(1).$$

(b) **Linear equivalence**:

$$D' \sim D \implies h_{D'} = h_D + O(1).$$

(c) **Functoriality**:

$$\varphi : X \to Y \implies h_{X,\varphi^*D}(P) = h_{Y,D}\bigl(\varphi(P)\bigr) + O(1).$$

(d) **Additivity**:

$$h_{D_1+D_2} = h_{D_1} + h_{D_2} + O(1).$$

"(b), (c), (d) *convert geometry into arithmetic*"

(e) **Finiteness** (Northcott Property): $\underline{D \text{ ample}}$.

$$\bigl\{P \in X(K) : h_D(P) \le B\bigr\} \text{ is finite.}$$

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## Canonical Height: Construction

$A/K$ is an abelian variety

**Def**: $D$ is *symmetric* if $[-1]^*D \sim D$.
$D$ is *anti-symmetric* if $[-1]^*D \sim -D$.

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Abelian varieties satisfy all sorts of interesting divisor relations, including:

$$[m]^*D \sim \frac{m^2+m}{2}D + \frac{m^2-m}{2}[-1]^*D = \begin{cases} m^2D & \text{if } D \text{ is symmetric,} \\ mD & \text{if } D \text{ is anti-symmetric.} \end{cases}$$

Thus for symmetric $D$, get

$$\begin{aligned} h_D\bigl([m]P\bigr) &= h_{[m]^*D}(P) + O(1) && \text{functoriality,} \\ &= h_{m^2D}(P) + O(1) && \text{linear equivalence,} \\ &= m^2h_D(P) + O(1) && \text{additivity.} \qquad (*) \end{aligned}$$

*Intuition*: $[m]P$ is about $m^2$ as complicated as $P$.
*Observation*: That $O(1)$ is annoying! ☺

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**Theorem** (Néron–Tate): $D \in \operatorname{Div}(A)$ symmetric.

(a) $\hat{h}_D(P) := \lim_{n\to\infty} \dfrac{1}{4^n} h_D\bigl([2^n]P\bigr)$ converges.

(b) $\hat{h}_D(P) = h_D(P) + O(1)$ for $P \in A(K)$.

(c) $\hat{h}_D\bigl([m]P\bigr) = m^2 \cdot \hat{h}_D(P)$ for $P \in A(K)$, $m \in \mathbb{Z}$.

(d) $D' \sim D \implies \hat{h}_{D'} = \hat{h}_D$.

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*Proof*. (a) **Claim**:

$$\left(\frac{1}{4^n}h_D\bigl([2^n]P\bigr)\right)_{n\ge 0}\text{is Cauchy, hence converges.}$$

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From $(*)$ with $m=2$ and $P \leftarrow [2^i]P$, we get

$$h_D\bigl([2^{i+1}]P\bigr) = 4h_D\bigl([2^i]P\bigr) + O(1). \qquad (**)$$

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Telescoping sum argument:

$$\begin{aligned}
&\left|\frac{1}{4^n}h_D\bigl([2^n]P\bigr) - \frac{1}{4^k}h_D\bigl([2^k]P\bigr)\right| \\
&\quad = \left|\sum_{i=k}^{n-1}\left(\frac{1}{4^{i+1}}h_D\bigl([2^{i+1}]P\bigr) - \frac{1}{4^i}h_D\bigl([2^i]P\bigr)\right)\right| \quad \text{telescoping sum,} \\
&\quad \le \sum_{i=k}^{n-1}\frac{1}{4^{i+1}}\left|h_D\bigl([2^{i+1}]P\bigr) - 4h_D\bigl([2^i]P\bigr)\right| \quad \text{triangle inequality,} \\
&\quad \le \sum_{i=k}^{n-1}\frac{1}{4^{i+1}}\cdot C \quad \text{from } (**), \\
&\quad \le \frac{1}{4^k}\cdot\frac{C}{3} \longrightarrow 0 \quad \text{as } n \ge k \to \infty. \qquad (***)
\end{aligned}$$

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(b) In $(***)$, taking

$$k = 0 \quad \text{and let} \quad n \to \infty$$

yields

$$\left|\hat{h}_D(P) - h_D(P)\right| \le \frac{C}{3}.$$

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(c) We compute

$$\begin{aligned}\hat{h}_D\big([m]P\big) &:= \lim_{n\to\infty} 4^{-n}\cdot h_D\big([2^n m]P\big)\\ &= \lim_{n\to\infty} 4^{-n}\cdot \Big(m^2 h_D\big([2^n]P\big) + \overbrace{O(1)}^{\text{Independent of } n \text{ and } P.}\Big)\\ &= m^2\hat{h}_D(P).\end{aligned}$$

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(d) Follows from $h_D = h_{D'} + O(1)$.

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**Remark**. There is a similar construction for anti-symmetric divisors, and then for general divisors, by writing

$$D = \underbrace{\frac{D+[-1]^*D}{2}}_{\text{symmetric}} + \underbrace{\frac{D-[-1]^*D}{2}}_{\text{anti-symmetric}}.$$

We will (mostly) use symmetric divisors.

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**Theorem** (Néron–Tate): $D \in \operatorname{Div}(A)$ ample and symmetric.
(a) $\hat{h}_D : A(K) \to \mathbb{R}$ is a quadratic form.
(b) $\hat{h}_D(P) \ge 0$ for all $P \in A(K)$.
(c) $\hat{h}_D(P) = 0$ if and only if $P \in A(K)_{\text{tors}}$.
(d) $\hat{h}_D$ extends to a positive definite quadratic form on

$$A(K)\otimes\mathbb{R} \cong \mathbb{R}^{\operatorname{rank} A(K)}.$$

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Proof sketch at end if there's time, but first:

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**Definition**. The *canonical* (*Néron–Tate*) *height pairing* is

$$\langle \cdot\,, \cdot \rangle_D : A(\bar{K}) \times A(\bar{K}) \longrightarrow \mathbb{R},$$
$$\langle P, Q\rangle_D = \frac{1}{2}\Bigl(\hat{h}_D(P+Q) - \hat{h}_D(P) - \hat{h}_D(Q)\Bigr).$$

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**Definition**. The *Néron–Tate regulator of*

$$P_1, \ldots, P_r \in A(K)$$

is

$$\operatorname{Reg}_D(P_1, \ldots, P_r) = \det\Bigl(\langle P_i, P_j\rangle_D\Bigr)_{1\le i,j\le r}.$$

We also let

$$\operatorname{Reg}_D(A/K) = \operatorname{Reg}_D(P_1, \ldots, P_r)$$

for any basis $P_1, \ldots, P_r$ of $A(K)/A(K)_{\text{tors}}$.

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**Remark**. $\hat{h}_D$ makes $A(K) \otimes \mathbb{R}$ into a Euclidean space, where the norm

$$\|P\|_D := \sqrt{\hat{h}_D(P)}$$

measures of the arithmetic complexity of the point $P \in A(K)$, where we can measure "complexity" angles

$$\langle P, Q\rangle_D := \|P\|_D \cdot \|Q\|_d \cdot \cos\theta_D(P, Q),$$

and where $\operatorname{Reg}_D(A/K)$ is the co-volume of the lattice

$$A(K)/A(K)_{\text{tors}} \cong \mathbb{Z}^{\operatorname{rank} A(K)} \quad \text{sitting in}$$
$$\text{the Euclidean space} \quad A(K) \otimes \mathbb{R} \cong \mathbb{R}^{\operatorname{rank} A(K)}.$$

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*Proof Sketch.* [As Time Allows]
We'll start with (b) and (c), since they tie in with Lecture #3.

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(b) $[\hat{h}_D(P) \ge 0]$

$$D \text{ ample} \implies h_D(Q) \ge -C \quad \text{for all } Q \in A(K), \text{ by positivity,}$$

$$\implies \hat{h}_D(P) = \lim_{n\to\infty} \frac{1}{4^n} h_D\bigl([2^n]P\bigr) \ge \lim_{n\to\infty} \frac{1}{4^n} \cdot (-C) = 0.$$

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(c) $\Bigl[P \in A(K)_{\text{tors}} \implies \hat{h}_D(P) = 0\Bigr]$

$$P \in A(K)_{\text{tors}} \implies \hat{h}_D(P) := \lim_{n\to\infty} 4^{-n} h_D\bigl(\underbrace{[2^n]P}_{\text{Only finitely many possible values}}\bigr) = 0.$$

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(c) $\Bigl[\hat{h}_D(P) = 0 \implies P \in A(K)_{\text{tors}}\Bigr]$

$\hat{h}_D(P) = 0$

$\implies \hat{h}_D\bigl([m]P\bigr) = m^2 \hat{h}_D(P) = 0$ for all $m \ge 1$,

$\implies h_D\bigl([m]P\bigr) \le C$ with $C$ independent of $m$, using $\hat{h}_D = h_D + O(1)$,

$$\implies \bigl\{[m]P : m \ge 1\bigr\} \subseteq \underbrace{\bigl\{Q \in A(K) : h_D(Q) \le C\bigr\}}_{\text{Finite set, since } D \text{ is ample.}}$$

$\implies [m_1]P = [m_2]P$ for some $m_1 > m_2 \ge 1$ (pigeonhole principle),

$\implies P \in A(K)_{\text{tors}}$.

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(a) [$\hat{h}_D(P)$ is a quadratic form on $A(K)$]
This is a computation using the Theorem of the Cube. See the lecture notes.

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(d) [$\hat{h}_D(P)$ is positive definite on $A(K) \otimes \mathbb{R}$]
The positive definiteness on $A(K)/A(K)_{\mathrm{tors}}$ follows from (b) & (c). But more is needed for $A(K) \otimes \mathbb{R}$, with the Northcott property playing a crucial role.