The Torelli locus and Newton polygons
AWS 2024: Lecture Notes

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February 23, 2024

Contents

1 Introduction **7**
1.1 The Torelli locus . . . 7
1.2 The boundary . . . 8
1.3 Arithmetic invariants . . . 9
1.3.1 Outline of the lecture notes . . . 11

2 The Torelli locus **13**
2.1 Overview . . . 13
2.2 Background on abelian varieties . . . 14
2.2.1 Complex tori . . . 14
2.2.2 Complex abelian varieties . . . 14
2.2.3 Polarized abelian varieties, with a sympletic basis . . . 15
2.2.4 Moduli spaces of abelian varieties . . . 15
2.2.5 Algebraic definition of abelian varieties . . . 16
2.3 Background on curves . . . 16
2.3.1 Curves . . . 16
2.3.2 Curves with automorphisms . . . 17
2.3.3 Holomorphic 1-forms and the genus . . . 17
2.3.4 The Riemann–Hurwitz formula . . . 18
2.3.5 Moduli spaces of curves . . . 18
2.4 Background on the Torelli map . . . 20
2.4.1 The Jacobian . . . 20
2.4.2 The Picard group . . . 20
2.4.3 The Abel–Jacobi map . . . 21
2.4.4 Variations on the Abel–Jacobi map . . . 21
2.4.5 Torelli's Theorem . . . 22
2.4.6 The Torelli morphism . . . 22
2.5 Related results . . . 22
2.5.1 Compactifications . . . 22
2.5.2 A stacky perspective . . . 23
2.5.3 The Schottky problem . . . 23
2.6 Open questions . . . 23

3 Arithmetic Invariants 25
3.1 Overview . . . 25
3.1.1 Collapsing of p-torsion points modulo p . . . 25
3.1.2 Supersingular elliptic curves . . . 26
3.1.3 Ordinary and supersingular elliptic curves . . . 27
3.2 Background . . . 27
3.2.1 The p-torsion group scheme . . . 27
3.2.2 The p-rank and a-number . . . 28
3.2.3 The p-divisible group . . . 28
3.2.4 The Newton polygon . . . 29
3.2.5 The Newton polygon, version 2 . . . 30
3.2.6 Dieudonné modules . . . 30
3.2.7 The Ekedahl-Oort type . . . 31
3.3 Main theorems . . . 32
3.3.1 The difference between p-rank 0 and supersingular . . . 32
3.4 Related results . . . 33
3.4.1 Examples for low dimension . . . 33
3.5 Open questions . . . 33

4 Existence of curves with given invariants 35
4.1 Overview . . . 35
4.2 Background . . . 35
4.2.1 The Newton polygon of a curve . . . 35
4.2.2 Computing the zeta function . . . 37
4.2.3 The Hasse–Witt and the Cartier–Manin matrices . . . 37
4.2.4 The de Rham cohomology . . . 38
4.3 Main theorems . . . 39
4.3.1 Small genus . . . 39
4.4 Related results . . . 40
4.4.1 Hermitian curves are supersingular . . . 40
4.4.2 Non-existence of superspecial curves . . . 40
4.4.3 Artin–Schreier curves . . . 41
4.5 Open questions . . . 43
4.5.1 Supersingular curves . . . 43
4.5.2 Counting the number of non-ordinary curves . . . 43
4.5.3 Double covers of an elliptic curve . . . 43

5 Complete subvarieties 45
5.1 Overview . . . 45
5.2 Background: The boundary of $\mathcal{M}_g$ . . . 45
5.2.1 Clutching maps . . . 45
5.2.2 Components of the boundary . . . 46
5.3 Main theorems: Complete subvarieties . . . 47
5.4 Related results . . . 47
5.5 Open questions: complete subvarieties . . . 47

6 Intersection **49**
6.1 Overview . . . 49
6.2 Background . . . 50
6.2.1 Specialization and purity . . . 50
6.2.2 Notation for the strata . . . 51
6.2.3 Dimensions of the strata . . . 51
6.2.4 Unlikely intersections . . . 52
6.3 Main theorems . . . 52
6.3.1 Invariants of stable curves . . . 53
6.3.2 A geometric proof for supersingular genus 4 curves . . . 53
6.3.3 Results about the p-rank stratification . . . 54
6.3.4 Increasing the p-rank . . . 55
6.4 Related results . . . 56
6.5 Open questions . . . 57
6.5.1 Number of components of the strata . . . 57
6.5.2 A statistical approach . . . 57
6.5.3 Intersection of the supersingular locus with the boundary . . . 58
6.5.4 Double covers of an elliptic curve . . . 58

7 Curves and abelian varieties with cyclic action **59**
7.1 Overview . . . 59
7.2 Background . . . 59
7.2.1 Equations of cyclic covers of the projective line . . . 59
7.2.2 The genus and the signature . . . 61
7.2.3 Hurwitz spaces . . . 61
7.3 Main theorems . . . 61
7.4 Related results - Shimura varieties . . . 63
7.4.1 Shimura datum for the moduli space of abelian varieties . . . 63
7.4.2 Shimura data of PEL-type . . . 63
7.4.3 Shimura subvariety attached to a monodromy datum . . . 63
7.5 Open questions . . . 64

8 Newton polygons for cyclic actions **65**
8.1 Overview . . . 65
8.2 Background . . . 65
8.2.1 Abelian varieties with complex multiplication . . . 65
8.2.2 Constraints on the invariants . . . 65
8.3 Main theorems . . . 66
8.4 Related results . . . 67
8.4.1 Inductive results . . . 67
8.4.2 Curves that are not μ-ordinary . . . 67
8.4.3 Other references . . . 68
8.5 Open questions . . . 68
8.5.1 Newton polygons on special abelian families . . . 68
8.5.2 Field of definition . . . 68

9 Projects **69**
9.1 Problems in characteristic zero . . . 69
9.2 Counting non-ordinary curves . . . 70
9.2.1 The question . . . 70
9.3 Supersingular curves in special families . . . 70
9.3.1 Supersingular curves in two-dimensional special families . . . 70
9.3.2 Field of definition of supersingular curves in special families . . . 71
9.3.3 Special families of non-cyclic covers . . . 71
9.4 Double covers . . . 72
9.4.1 The p-ranks of double covers of an elliptic curve . . . 72
9.4.2 Non-ordinary curves in complete families of $\mathcal{M}_g$ . . . 72
9.5 The geometry of the supersingular locus . . . 73
9.6 Problem about Ekedahl–Oort strata . . . 74

Chapter 1

Introduction

This lecture series is about the Torelli locus in the moduli space of abelian varieties, with applications to Newton polygons of curves in positive characteristic. In general, the lectures will cover two topics: the first is about the *geometry* of the Torelli locus; the second is about the *arithmetic* invariants of abelian varieties that occur for Jacobians of smooth curves in positive characteristic.

This is the draft of this document that we will use for the Arizona Winter School in March 2024. I am still planning to make improvements at a later time, so comments are welcome.

I'd like to thank these people for their support and helpful suggestions about this document: Jeff Achter, Dusan Dragutinović, Steven Groen, Valentijn Karemaker, and Soumya Sankar. Also thanks to the NSF for their partial support (DMS-22-00418).

1.1 The Torelli locus

Let g be a positive integer. Suppose X is a (smooth, projective, connected) curve of genus g. The Jacobian J_X of X represents the quotient of the group of divisors of degree zero by the subgroup of principal divisors. One can show that the Jacobian J_X is a (principally polarized) abelian variety of dimension g. Many facts about X are determined by its Jacobian; for example, the unramified cyclic degree ℓ covers of X are determined by ℓ-torsion points on the Jacobian J_X.

For $1 \leq g \leq 3$, almost every principally polarized abelian variety is a Jacobian. For example, a p.p. abelian variety of dimension $g = 1$ is an elliptic curve. A p.p. abelian surface (resp. threefold) is the Jacobian of a smooth curve of genus 2 (resp. 3) unless it decomposes as a product, together with the product polarization.

For $g \geq 4$, the situation is more interesting because not every principally polarized abelian variety is a Jacobian. There are several methods to determine which p.p. abelian varieties are Jacobians but these are fairly difficult. It is often possible to study Jacobians of curves in a more explicit and concrete way than for a typical abelian variety. On the other hand, there are techniques for studying families of abelian varieties that do not apply when studying families of Jacobians of curves. This leads to a very valuable and rewarding exchange between these topics.

Consider the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. Within $\mathcal{A}_g$, we can consider the Torelli locus whose points represent Jacobians of curves. This sublocus of $\mathcal{A}_g$ has essential importance and plays an important role in many problems. Let $\mathcal{M}_g$ denote the moduli space of (smooth, projective, connected) curves of genus g. For $r \geq 1$, we also consider $\mathcal{M}_{g;r}$, the moduli space of curves of genus g together with r marked points.

The Torelli morphism $\tau : \mathcal{M}_g \to \mathcal{A}_g$ takes a curve X to its Jacobian. It is an embedding, meaning that X is uniquely determined by J_X. The *open Torelli locus* $\mathcal{T}_g^\circ$ is the image of τ; it is the locus of all principally polarized abelian varieties of dimension g that are Jacobians.

When $g = 1, 2, 3$, then $\mathcal{T}_g^\circ$ is open and dense in $\mathcal{A}_g$, meaning that almost every principally polarized abelian variety of dimension $g \leq 3$ is a Jacobian. For $g \geq 2$, the dimension of $\mathcal{M}_g$ is $3g - 3$, while the dimension of $\mathcal{A}_g$ is $g(g + 1)/2$. So, as g increases, the open Torelli locus has increasingly high codimension in $\mathcal{A}_g$.

1.2 The boundary

Surprisingly, some facts about smooth curves can be proven using singular curves; some facts about principally polarized abelian varieties that are indecomposable can be proven using principally polarized abelian varieties that decompose. For this reason, it is useful to consider compactifications of these moduli spaces, namely the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ and a toroidal compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$.

The points of the boundary of $\mathcal{M}_g$ represent stable singular curves, which are either of compact or non-compact type. When the dual graph of a curve is a tree, we say that the curve has compact type. To construct a singular curve of compact type, we take two curves (which are smooth, or of compact type); we choose a point on each, and identify these points in an ordinary double point. If $g_1 + g_2 = g$, this yields a morphism:

$$\kappa_{g_1,g_2} : \overline{\mathcal{M}}_{g_1;1} \times \overline{\mathcal{M}}_{g_2;1} \to \overline{\mathcal{M}}_g.$$

The Jacobian of a singular curve of compact type is an abelian variety, although it does decompose together with the product polarization.

To construct a singular curve of non-compact type, we take a curve, choose two points on it, and identify these in an ordinary double point. This yields a morphism:

$$\kappa_0 : \overline{\mathcal{M}}_{g-1;2} \to \overline{\mathcal{M}}_g.$$

The Jacobian of a singular curve of non-compact type is a semi-abelian variety. Later notes will include more description of semi-abelian varieties, including the toric rank of a semi-abelian variety and the toroidal compactification $\tilde{A}_g$.

Historically, many statements about the geometry of $\mathcal{M}_g$ use the morphisms κ_{g_1,g_2}, κ_0, which are called clutching morphisms. The Torelli map extends to a map $\overline{\tau} : \overline{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g$. However, $\overline{\tau}$ is no longer an embedding; in fact, some of its fibers have positive dimension.

1.3 Arithmetic invariants

Let k be an algebraically closed field of positive characteristic p. An elliptic curve over k can be ordinary or supersingular. We say that an elliptic curve is ordinary if it has point of order p; alternatively, an elliptic curve is ordinary if its Newton polygon has slopes of zero and one. Otherwise, the elliptic curve is supersingular. There are many results about ordinary and supersingular elliptic curves, due to Deuring [Deu41] and Igusa [Igu58]; for example, for a fixed prime p, most elliptic curves are ordinary and the number of isomorphism classes of supersingular elliptic curves is approximately $p/12$. See also [Man61].

For a p.p. abelian variety A defined over k, the action of Frobenius determines important information. To keep track of this information, there are combinatorial invariants called the p-rank, the Newton polygon, the Ekedahl–Oort type, and the a-number. The p-rank is the integer f such that the number of p-torsion points on A equals p^f. The Newton polygon is determined by the characteristic polynomial of Frobenius on the crystalline cohomology; when $A = J_X$ for a curve X defined over a finite field $\mathbb{F}$, the Newton polygon keeps track of the number of points on X defined over finite extensions of $\mathbb{F}$. The Ekedahl–Oort type is an invariant that classifies the structure of the p-torsion group scheme $A[p]$ of A; when $A = J_X$, this is the same as the structure of the de Rham cohomology as a module under Frobenius F and Verschiebung V. The a-number is the number of generators of $A[p]$ as a module under F and V.

The possibilities for the Newton polygon and Ekedahl–Oort type of a p.p abelian variety are well understood. In contrast, in most cases it is not known which Newton polygons and Ekedahl–Oort types occur for Jacobians of curves for a given prime p. Some Newton polygons and Ekedahl–Oort types have been shown to occur for Jacobians and some Ekedahl–Oort types have been ruled out. More generally, the stratifications of $\mathcal{A}_g$ by these invariants are well understood; however, it is not understood how these stratifications intersect the Torelli locus. As applications of the theory covered in this lecture series, I will show how the geometric techniques used to study moduli spaces can shed light on these questions.

Lectures:

Here is a tentative schedule of lectures. These lectures are about abelian varieties defined over an algebraically closed field. The first half of each lecture includes material that makes sense for fields of any characteristic; the second half of each lecture includes applications for abelian varieties in positive characteristic.

1. **The Torelli locus and arithmetic invariants**

   In the first half of this lecture, I will give several descriptions of the Torelli locus in the moduli space $\mathcal{A}_g$ of abelian varieties of dimension g. With a dimension count, we can see that the Torelli locus is open and dense inside $\mathcal{A}_g$ when $1 \leq g \leq 3$, and has positive codimension for $g \geq 4$.

   In the second half of this lecture, I will describe some arithmetic invariants of abelian varieties in positive characteristic p. These include: the p-rank, the Newton polygon, the Ekedahl–Oort type, and the a-number, see [Pri19] for a survey. As some applications, we can see the proofs of these facts, for every prime p:
   (i) there exists an ordinary smooth curve of every genus g, [Mil72];

(ii) there exists a non-ordinary smooth curve of every genus g; and
(iii) there exists a supersingular curve of genus 2 [Ser83], [IKO86].

The proofs make use of the Cartier operator.

2. The boundary of the moduli spaces of curves and abelian varieties

In the first half of this lecture, I will describe the boundary of the moduli space of curves and the clutching morphisms, as described in Section 5.2. The boundary is the image of the clutching morphisms, whose domain consists of products of moduli spaces of curves with marked points. Then we will cover some results of Diaz [Dia84] and Looijenga [Loo95a] that show that a subspace $S \subset \overline{\mathcal{M}}_g$ having codimension at most g must intersect the boundary.

In the second half of this lecture, I will describe the purity result of de Jong and Oort [dJO00a] for the Newton polygon stratification of $\mathcal{A}_g$. As an application, for every prime p, this yields a proof that there exists a supersingular curve of genus 3 [Oor91a], and a supersingular curve of genus 4 [KHS20], [Pri]. We will see that this proof does not extend to curves of higher genus. I will also explain how the boundary technique can be used to study the p-rank stratification of $\mathcal{M}_g$ [FvdG04].

3. Special families of abelian varieties

In the first half of this lecture, I will describe the situation for abelian varieties having additional structure; namely, whose automorphism group contains a cyclic group. The moduli spaces of these provide examples of Deligne–Mostow Shimura varieties. We say this moduli space is *special* if an open and dense subset of a component of the Shimura variety is contained in the Torelli locus. In particular, we consider families of Jacobians of curves that are cyclic covers of the projective line. The families that have special moduli spaces were classified by Moonen [Moo10]. The situation for Jacobians of abelian covers of the projective line is not fully understood and is related to a conjecture of Coleman and Oort.

In the second half of this lecture, I will describe constraints on the Newton polygon and Ekedahl–Oort type of an abelian variety in these special families. As an application, this shows that there exist supersingular curves of genus 5, 6, and 7, under certain congruence conditions on the prime p [LMPT19]. Furthermore, I will describe the rate of growth of the number of non-ordinary curves in these families [CP].

4. Torsion points and unramified covers

In the first part of this lecture, I will describe the correspondence between ℓ-torsion points on the Jacobian of a curve C and unramified $\mathbb{Z}/\ell\mathbb{Z}$-covers of C. In the second half of this lecture, we will see how the p-torsion and the ℓ-torsion on Jacobians are independent of each other, in a way that can be made precise using ℓ-adic monodromy groups of the p-rank stratification [AP08].

1.3.1 Outline of the lecture notes

Three of these chapters are written for abelian varieties and curves over any algebraically closed field, such as $\mathbb{C}$; these are Chapters 2, 5, and 7. The other chapters are about invariants that are defined only in positive characteristic.

Chapter 2

The Torelli locus

2.1 Overview

The main focus of these talks is the Torelli locus $\mathcal{T}_g$ within the moduli space $\mathcal{A}_g$ of principally polarized (p.p.) abelian varieties of dimension $g \geq 1$.

In writing (or reading) this chapter, there is a basic dilemma. It is important to start with a good foundation. On the other hand, with limitations on time and space, it is not possible to improve on references such as these books (and others):

Analytic theory of abelian varieties by Swinnerton-Dyer, [SD74];
Abelian varieties by Mumford [Mum08]
Abelian varieties by Milne, [Mil];
Complex abelian varieties by Birkenhake and Lange, [BL04];
Abelian varieties by Lange [Lan23]
Abelian varieties (preliminary version) by Edixhoven, van der Geer, and Moonen, [EvdGM]
Curves and their Jacobians by Mumford [Mum75]
Geometry of algebraic curves by Arbarello, Cornalba, Griffiths, Harris [ACGH85], [ACG11].
Algebraic curves and Riemann surfaces by Miranda [Mir95].
Moduli of Curves by Harris and Morrison [HM98]

In addition, most of these books were written with a complex analytic viewpoint, which provides a lot of intuition but which is not sufficient for many of the topics in the later chapters. In this chapter, we work over $k = \mathbb{C}$, although much of the content also applies for any algebraically closed field k.

So, the goal for this chapter is modest: to introduce the main concepts, so that we can continue with the key themes of the lecture series. The main concepts are:

The Jacobian of a curve of genus g is a p.p. abelian variety of dimension g.

The Torelli morphism maps the moduli space $\mathcal{M}_g$ of curves of genus g into the moduli space $\mathcal{A}_g$ of p.p. abelian varieties of dimension g. This map is injective on k-points.

The dimension of $\mathcal{A}_g$ is $g(g+1)/2$ and the dimension of $\mathcal{M}_g$ is $3g-3$ (for $g \geq 2$). This implies that most p.p. abelian varieties of dimension $g \geq 4$ are not Jacobians.

At a later time, I will return to this chapter to expand on the most important aspects. and add additional examples, citations, and precision.

2.2 Background on abelian varieties

There is a lot of foundational material here. It may be difficult to absorb it all on the first reading. It may be helpful to focus on the examples.

We follow [BL04, Chapters 4,8,11].

Let $g \geq 1$ be an integer. We denote complex conjugation with an overline.

2.2.1 Complex tori

Example 2.2.1. A complex torus of dimension 1 is isomorphic to $\mathbb{C}/\Lambda$ where Λ is a lattice. After adjusting by the action of $\mathrm{SL}_2(\mathbb{Z})$, we can suppose Λ is generated by 1 and τ, where τ is in the upper half plane $\mathfrak{h}$. The Hermitian form $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is given by $H(v, w) = v \cdot \overline{w}/\mathrm{Im}(\tau)$. This is a positive definite form.

More generally, consider a complex torus $X = V/\Lambda$ where V is a complex vector space of dimension g and Λ is a lattice. We choose a $\mathbb{Z}$-basis $\lambda_1, \ldots, \lambda_{2g}$ for Λ in terms of a basis $e_1, \ldots, e_g$ for V. Writing the former in terms of the latter gives a $g \times 2g$-matrix Π called the *period matrix.*

Proposition 2.2.2. *[BL04, Proposition 1.1.2] A $g \times 2g$-matrix Π is the period matrix of a complex torus if and only if the $2g \times 2g$-matrix $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ is invertible.*

2.2.2 Complex abelian varieties

A good reference for complex abelian varieties is Birkenhake and Lange [BL04, Chapter 4]. See also [Mum08].

Definition 2.2.3. A complex abelian variety is a complex torus admitting an ample line bundle.

Suppose $X = V/\Lambda$ is a complex torus. Then X is a projective complex analytic space, and thus a projective complex algebraic variety.

The condition of having an ample line bundle can be described in several different ways. First, here are the Riemann relations.

Theorem 2.2.4. *[BL04, Theorem 4.2.1] The complex torus $\mathbb{C}^g/\Pi\mathbb{Z}^{2g}$ is an abelian variety if and only if there exists a non-degenerate $2g \times 2g$ alternating matrix A such that the following* Riemann relations *are true:*

(i) $\Pi A^{-1T}\Pi = 0$*; and*

(ii) $i\Pi A^{-1T}\overline{\Pi} > 0$.

In this context, A is the matrix of the alternating form E defining the polarization.

The second interpretation involves Hermitian forms. A *Hermitian form* on V is a map $H : V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first argument and such that $H(v, w) = \overline{H(w, v)}$ for all $v, w \in V$. A Hermitian form is *positive semi-definite* if $H(v, v) \geq 0$ for all $v \in V$; it is positive definite if it is positive semi-definite and $H(v, v) = 0$ if and only if $v = 0$; it is *non-degenerate* if $H(u, v) = 0$ for all $v \in V$ implies $u = 0$.

Definition 2.2.5. A *Riemann form* on $X = V/\Lambda$ is a positive definite non-degenerate Hermitian form H on V such that the restriction of $E = \text{Imaginary}(H)$ to Λ is integer valued.

Theorem 2.2.6. *A complex torus is isomorphic to an abelian variety X over $\mathbb{C}$ if and only if it has a Riemann form.*

A third interpretation is as follows. Suppose $X = V/\Lambda$ is a complex torus and let X^* be its dual. Let $\overline{\Omega} = \text{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ be the vector space of $\mathbb{C}$-antilinear forms. Given an analytic representation $F : V \to \overline{\Omega}$, consider the form $F : V \times V \to \mathbb{C}$ given by $(v, w) \mapsto F(v)(w)$. A *polarization* is an isogeny $X \to X^*$ whose analytic representation is a positive definite Hermitian form. A *principal polarization* is a polarization that is an isomorphism.

In [BL04, Section 2.4], there is a description of how a line bundle L on X determines a map $\phi_L : X \to X^*$; it is an isogeny if and only if L is ample. Conversely, by [BL04, Theorem 2.5.5], if $X = V/\Lambda$ is a complex torus and $\phi : X \to X^*$ is a polarization, then X is an abelian variety.

2.2.3 Polarized abelian varieties, with a sympletic basis

This next part will be important for defining the Siegel upper half space.

Suppose $X = V/\Lambda$ is a p.p. abelian variety of dimension g and H is a Hermitian form defining a principal polarization. We choose a symplectic $\mathbb{R}$-basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ of Λ for H; this means that $H(\lambda_i, \mu_j) = \delta_{i,j}$. The vectors $\mu_1, \ldots, \mu_g$ form a $\mathbb{C}$-basis for V. The alternating form $E = \text{Im}(H)$ is given by the matrix $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$, with respect to this basis. The period matrix is given by $\Pi = (Z, I_g)$ for some $g \times g$ matrix Z.

Proposition 2.2.7. *[BL04, Proposition 8.1.1] (a) ${}^T Z = Z$ and $\text{Im}(Z) > 0$; and*
(b) $(\text{Im}(Z))^{-1}$ is the matrix of H with respect to the basis $\mu_1, \ldots, \mu_g$.

2.2.4 Moduli spaces of abelian varieties

Let $\mathcal{A}_{g,\mathbb{C}}$ be the moduli space of complex p.p. abelian varieties of dimension g.

Example 2.2.8. Abelian varieties of dimension $g = 1$ are parametrized by $\tau \in \mathfrak{h}$, up to the action of $\text{SL}_2(\mathbb{Z})$. The condition of having a principal polarization is automatically satisfied. This shows that $\dim(\mathcal{A}_1) = 1$.

We follow [BL04, Chapter 8]. Recall the material in Section 2.2.3.

Definition 2.2.9. The Siegel upper half space $\mathfrak{h}_g$ is the set of $g \times g$ complex-valued matrices satisfying ${}^T Z = Z$ and $\text{Im}(Z) > 0$.

Then $\mathfrak{h}_g$ has dimension $g(g+1)/2$ because it is an open submanifold of the vector space of symmetric $g \times g$ matrices. By [BL04, Proposition 8.1.2], $\mathfrak{h}_g$ is a moduli space for principally polarized abelian varieties with symplectic basis. By [BL04, Theorem 8.2.6], $\mathcal{A}_g$ is a quotient of $\mathfrak{h}_g$ by the sympletic group $\text{Sp}_{2g}(\mathbb{Z})$. This shows the following.

Theorem 2.2.10. *The moduli space $\mathcal{A}_g$ is irreducible and has dimension $g(g + 1)/2$.*

See [MFK94] by Mumford, Fogarty, and Kirwan for some other constructions of $\mathcal{A}_g$.

2.2.5 Algebraic definition of abelian varieties

A complex torus is an abelian variety if and only if it is an algebraic variety. In this section, we give a fully algebraic definition.

Definition 2.2.11. An *abelian variety* is a smooth irreducible projective algebraic variety X that is also a group. This means that it has a group law $m : X \times X \to X$ and both m and the inverse map are morphisms. A *principal polarization* is an isomorphism $X \to X^*$, satisfying an additional property.

2.3 Background on curves

We work over an algebraically closed field k.

2.3.1 Curves

Definition 2.3.1. A curve is a connected projective variety of dimension 1.

Example 2.3.2. Let $\mathbb{P}^1$ denote the projective line. This is the unique curve of genus 0. An elliptic curve is given by the vanishing of a smooth cubic in $\mathbb{P}^2$.

The easiest way to describe a curve of positive genus is with an affine equation. Frequently, we consider an affine curve $C' \subset \mathbb{A}^2$ given by the vanishing of a polynomial equation $h(x, y) = 0$. It is no loss of generality to work with affine curves because of this fact:

Fact 2.3.3. *For every affine curve $C' \subset \mathbb{A}^2$, there exists a unique smooth projective curve C such that $C' \subset C$.*

Sometimes, the curve C can be embedded in $\mathbb{P}^2$.

Example 2.3.4. Suppose $f(x) = x^3 + ax + b$ has distinct roots for some $a, b \in k$. (Here $p \neq 2, 3$). Consider the elliptic curve with affine equation $y^2 = f(x)$. It is the projective curve in $\mathbb{P}^2$ given by the vanishing of the homogeneous equation $y^2z = x^3 + axz^2 + bz^3$.

Sometimes, the curve C cannot be smoothly embedded in $\mathbb{P}^2$. Every curve can be smoothly embedded in $\mathbb{P}^3$, but this is not always helpful. It is often a hassle to find the equations that resolve the singularities of a curve. In light of Fact 2.3.3, we usually work with affine curves.

Example 2.3.5. Let C' be the curve with affine equation $y^2 = x^5 - 2x$ (here $p \neq 2, 5$). The homogenization $y^2z^3 = x^5 - 2xz^4$ has a singularity when $z = 0$. To find another affine patch for the curve that includes the points missing on this patch, we define $\bar{x} = 1/x$ and $\bar{y} = y\bar{x}^3$. The other affine patch is given by the affine equation $\bar{y}^2 = \bar{x} - 2\bar{x}^5$.

2.3.2 Curves with automorphisms

Definition 2.3.6. A *hyperelliptic curve* is a curve C that admits a cyclic cover $\pi : C \to \mathbb{P}^1$.

Fact 2.3.7. *If* $\mathrm{char}(k) \neq 2$*, a hyperelliptic curve has an affine equation* $y^2 = f(x)$ *for some separable polynomial* $f(x)$*. The hyperelliptic involution* ι *acts by* $\iota((x, y)) = (x, -y)$*. There is a unique hyperelliptic involution on a hyperelliptic curve* C *and it is contained in the center of the automorphism group of* C*.*

Definition 2.3.8. A *superelliptic curve* is a curve C that admits a cyclic cover $\pi : C \to \mathbb{P}^1$.

Fact 2.3.9. *If* $\mathrm{char}(k)$ *does not divide the degree* m *of* π*, then the superelliptic curve has an affine equation* $y^m = \prod_{i=1}^N (x - b_i)^{a_i}$*, with the following data:*

the degree of the cover is $m \geq 2$*;*

the number of branch points is $N \geq 3$*;*

the inertia type is a tuple $(a_1, \ldots, a_N)$ *with* $1 \leq a_i \leq m - 1$ *and* $\sum_{i=1}^N a_i \equiv 0 \bmod m$*;*

the branch points $\{b_1, \ldots, b_N\}$ *are a set of* N *distinct points in* $\mathbb{P}^1$*.*

Sometimes ∞ *is one of the branch points (say the last one); in which case the last term* $(x - b_N)^{a_N}$ *is removed from the equation.*

The μ_m*-action on* C *is given by* $\phi((x, y)) = (x, \zeta y)$ *for* $\zeta \in \mu_m$*.*

Definition 2.3.10. An *Artin–Schreier curve* is a curve C that admits a degree p cyclic cover $\pi : C \to \mathbb{P}^1$, where $p = \mathrm{char}(k)$.

Fact 2.3.11. *An Artin–Schreier curve has an affine equation* $y^p - y = h$ *for some* $h \in k(x)$*; the curve is connected if and only if* $h \neq z^p - z$ *for any rational function* $z \in k(x)$*. Without loss of generality, we can suppose that the order of the poles of* h *are relatively prime to* p*. The* $\mathbb{Z}/p\mathbb{Z}$*-action on* C *is given by* $\phi((x, y)) = (x, y + 1)$*. This cover is wildly ramified at each of the poles of* h*.*

2.3.3 Holomorphic 1-forms and the genus

Suppose C is a smooth projective curve. A 1-form ω is a smooth section of the cotangent bundle. The 1-form is *holomorphic* if it has no poles.

For a local description of ω near a point P, we consider a function z on an affine subset U of C containing P such that z vanishes with order 1 at P. Then ω has an expression of the form $f(z)dz$ where $f(z)$ is a rational function on U.

Example 2.3.12. The 1-form dx on $\mathbb{P}^1$ has a pole of order 2 at ∞. So $\mathrm{div}(dx) = -2[\infty]$.

For the elliptic curve $y^2 = x^3 + ax + b$ from Example 2.3.4, the 1-form dx/y is holomorphic.

Let Ω^1 denote the sheaf of 1-forms on C.

Definition 2.3.13. Let $H^0(C, \Omega^1)$ denote the vector space of holomorphic 1-forms. The *genus* g of C is the dimension of $H^0(C, \Omega^1)$.

Finding the orders of poles of a 1-form is a delicate process. The following lemma is useful.

Lemma 2.3.14. *[Mir95, IV, Lemma 2.6] Suppose $\pi : C_1 \to C_2$ is a cover of curves. If ω is a 1-form on C_2, then the pullback $\pi^*\omega$ is a 1-form on C_1. If π is not wildly ramified, and if $\eta \in C_1$ is a point, then* $\mathrm{ord}_\eta(\pi^*\omega) = (1 + \mathrm{ord}_{\pi(\eta)}(\omega))\mathrm{mult}_\eta(\pi) - 1$.

The following examples can be checked using Lemma 2.3.14.

Example 2.3.15. Let $p \neq 2$. Suppose $f(x)$ is a separable polynomial of degree $2g+1$ or $2g+2$. The hyperelliptic curve C with affine equation $y^2 = f(x)$ has genus g. A basis for $H^0(C, \Omega^1)$ is given by $\{dx/y, xdx/y, \ldots, x^{g-1}dx/y\}$.

Example 2.3.16. Consider the Artin–Schreier curve C with affine equation $y^p - y = h$ where $h \in k[x]$ is a polynomial of degree j and $p \nmid j$. Then the genus of C is $g = (p-1)(j-1)/2$. This can be proven with the wild Riemann–Hurwitz formula. A basis for $H^0(C, \Omega^1)$ is given by

$$\{y^r x^b dx \mid 0 \leq r \leq p-2,\ 0 \leq b \leq j-2,\ rj + bp \leq pj - j - p - 1\}.$$

2.3.4 The Riemann–Hurwitz formula

The Riemann–Hurwitz formula provides a good way to compute the genus.

Theorem 2.3.17. *(Riemann–Hurwitz formula) Suppose $\phi : C \to D$ is a degree d cover of curves. (If* $\mathrm{char}(k) > 0$*, assume the cover is tamely ramified.) For $\eta \in C$, let e_η denote the ramification index of ϕ at η. Then the genus g_C of C and the genus g_D of D are related by the formula:*

$$2g_C - 2 = d(2g_D - 2) + \sum_{\eta \in C}(e_\eta - 1).$$

Example 2.3.18. Let $p \nmid m$. Consider the superelliptic curve C with affine equation $y^m = \prod_{i=1}^N (x - b_i)^{a_i}$. Above the point $x = b_i$, the curve C has $g_i = \gcd(m, a_i)$ points, each with inertia group of order m/g_i. By the Riemann–Hurwitz formula, the genus of C satisfies:

$$2g_C - 2 = m(-2) + \sum_{i=1}^N g_i(\frac{m}{g_i} - 1).$$

In particular, if $g_i = 1$ for $1 \leq i \leq N$ (e.g., if m is prime), then $g_C = (N-2)(m-1)/2$.

2.3.5 Moduli spaces of curves

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. Let $\mathcal{H}_g$ be the moduli space of smooth hyperelliptic curves of genus g. In [MFK94], Mumford and Fogarty give three constructions of $\mathcal{M}_g$, using geometric invariant theory, covariants of points, and theta constants. The main goal of this section is to determine the dimensions of $\mathcal{M}_g$ and $\mathcal{H}_g$.

Let $n \geq 3$. Let P_n denote the space parametrizing unordered sets of n distinct points in $\mathbb{P}^1$, up to automorphisms of $\mathbb{P}^1$.

Proposition 2.3.19. *(See for example, [Mir95, page 213]) If $n \geq 3$, then* $\dim(P_n) = n - 3$.

Proof. There is a map $(\mathbb{P} - \{0, 1, \infty\})^{n-3} - \Delta_W \to P_k$, where Δ_W is the weak diagonal of tuples with repeated entries, where the map sends an ordered $n-3$ tuple $(x_1, \ldots, x_{n-3})$ to the set $\{0, 1, \infty, x_1, \ldots, x_{n-3}\}$. This map is surjective because of the triply transitive action of $\mathrm{Aut}(\mathbb{P}^1)$. It has finite fibers because there are only a finite number of ways to order a set of n points and only finitely many automorphisms sending the first three to 0, 1, and ∞. □

Corollary 2.3.20. *If $g \geq 1$, then $\dim(\mathcal{H}_g) = 2g - 1$.*

Proof. Every hyperelliptic curve of genus g is determined by its set of $2g + 2$ branch points. By Proposition 2.3.19, it follows that $\dim(\mathcal{H}_g) = 2g - 1$ for each $g \geq 1$. □

Theorem 2.3.21. *If $g \geq 2$, the moduli space $\mathcal{M}_g$ is irreducible and has dimension $3g - 3$. If $g = 1$, the moduli space $\mathcal{M}_{1;1}$ is irreducible and has dimension 1.*

For the irreducibility, see [DM69]. We sketch two proofs for the dimension.

Proof. (Sketch, following [Mir95, VII, Section 2])

Since every curve of genus 1 or 2 is hyperelliptic, Corollary 2.3.20 shows that $\dim(\mathcal{M}_{1;1}) = 1$ and $\dim(\mathcal{M}_2) = 3$.

Let $g \geq 3$. We consider extra data on a curve C of genus g and investigate the moduli spaces of these objects is turn. The proof makes extensive use of divisors, linear systems, and the Riemann–Roch theorem.

1. The data of (C, D), where D is a divisor of degree $2g - 1$.

   Every curve C of genus g has an effective divisor D of degree $2g - 1$. The number of parameters for this divisor is $2g - 1$. So it suffices to show that the number of parameters for (C, D) is $(3g - 3) + (2g - 1) = 5g - 4$.

2. The data of $(C, |D|)$ where $|D|$ is a complete linear system of degree $2g - 1$.

   We move from (C, D) to $(C, |D|)$ by taking D to its complete linear system $|D|$. Note that $\dim(|D|) = \deg(D) - g = g - 1$. So the number of parameters of the choice of an effective divisor E in $|D|$ is $g - 1$. So it suffices to show that the number of parameters for $(C, |D|)$ is $(5g - 4) - (g - 1) = 4g - 3$.

3. The data of (C, Q) where Q is a base-point free pencil of degree $2g - 1$.

   Given the complete linear system $|D|$ of degree $2g - 1$, we add the data of a pencil, or linear subspace, Q. Conversely, given a pencil Q, we can consider its complete linear system. Given $|D|$, the number of parameters for the choice of Q is the number of parameters for a line in a projective space of dimension $g - 1$. This is the dimension of the Grassmanian $\mathbb{G}(1, g - 1)$, which is $2g - 4$. So it suffices to show that the number of parameters for (C, Q) is $(4g - 3) + (2g - 4) = 6g - 7$.

4. The data of (C, F) where $F : C \to \mathbb{P}^1$ is a map of degree $2g - 1$, branched at $6g - 7$ points. The data for Q and F is equivalent, so it suffices to show that the number of parameters for (C, F) is $6g - 7$.

5. The data of $6g-7$ unordered points in $\mathbb{P}^1$.

   Given (C, F), we can forget all the data except for the unordered set of $6g-7$ branch points. Conversely, given a unordered set of $6g-7$ points, there are a non-zero finite number of maps $F : C \to \mathbb{P}^1$ of degree $2g-1$ that are branched at those points such that C has genus g. So it suffices to show that the number of parameters for the $6g-7$ points is $6g-4$, which we stated at the beginning of this remark.

□

Here is a sketch of another proof.

Proof. Let C be a complex analytic space. A direct cocycle calculation, as in Kodaira-Spencer theory, shows that first order deformations are parametrized by a subspace of $H^1(C, T_C)$, the first cohomology group with coefficients in the tangent sheaf. The same is true in the category of algebraic schemes.

For a curve C, then $\dim(C) = 1$. In this case, $H^2(C, T_C) = 0$, so deformations are unobstructed. Thus the deformation space of C is isomorphic to $H^1(C, T_C)$. Also T_C is the dual of the canonical bundle Ω_C. By the Riemann–Roch theorem, if $g \geq 2$, then $\dim(H^1(C, T_C)) = 3g-3$. □

2.4 Background on the Torelli map

2.4.1 The Jacobian

We loosely follow Miranda [Mir95, Chapter VIII], working over $\mathbb{C}$.

A *linear functional* is an element of the dual space $H^0(C, \Omega^1)^*$, namely a linear transformation $H^0(C, \Omega^1) \to \mathbb{C}$.

Loops c in C can be represented by homology classes. The homology group $H_1(C, \mathbb{Z})$ is a free abelian group of rank $2g$. Every homology class $[c]$ defines a linear functional $\int_{[c]} : H^0(C, \Omega^1) \to \mathbb{C}$, which takes a holomorphic 1-form ω to its integral over c. The linear functionals that occur in this way are called *periods*. The set Λ of periods is a subgroup of $H^0(C, \Omega^1)^*$.

Definition 2.4.1. The *Jacobian* of C is $\mathrm{Jac}(C) = H^0(C, \Omega^1)^*/\Lambda$.

By definition, $\mathrm{Jac}(C)$ is an abelian group. By choosing a basis for $H^0(C, \Omega^1)$, one can see that $\mathrm{Jac}(C) \cong \mathbb{C}^g/\Lambda$, which is a complex torus of dimension g. With additional work, one can show that the periods satisfy the Riemann relations. Thus there is a principal polarization on $\mathrm{Jac}(C)$. Thus $\mathrm{Jac}(C)$ is a principally polarized abelian variety.

2.4.2 The Picard group

Let $\mathrm{Div}(C)$ denote the group of divisors on C, namely finite sums of the form $D = \sum_{P \in C} n_P[P]$, where n_P is an integer for each point $P \in C$. The degree of D is $\sum_{P \in C} n_P$. The group $\mathrm{Div}(C)$ contains the subgroup $\mathrm{Div}^0(C)$ of divisors of degree 0.

A divisor D is *principal* if it is the divisor of a rational function f on C. This means that n_P is the order of vanishing of f at the point P. The degree of a principal divisor is 0. Let $\mathrm{PDiv}(C)$ be the set of principal divisors. Note that $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ and $\mathrm{div}(1/f) = -\mathrm{div}(f)$. This shows that $\mathrm{PDiv}(C)$ is a subgroup of $\mathrm{Div}^0(C)$.

Definition 2.4.2. The Picard group of C is $\mathrm{Pic}(C) = \mathrm{Div}(C)/\mathrm{PDiv}(C)$. Denote by $\mathrm{Pic}^0(C)$ the subgroup of $\mathrm{Pic}(C)$ given by classes of divisors of degree 0.

Remark 2.4.3. Another definition of the Jacobian is the connected component of the identity in the Picard group of divisors of degree 0.

2.4.3 The Abel–Jacobi map

Choose a base point $p_\circ$ on C. For each point $x \in C$, choose a path γ_x from $p_\circ$ to x. This is possible because C is connected (and this implies that $\mathrm{Pic}^0(C)$ is also connected). There is a map $C \to H^0(C, \Omega^1)^*$, sending x to the linear functional $\int_{\gamma_x}$ of integration along γ_x. This map is not well-defined because different paths from $p_\circ$ to x may not be homotopic. However, there is a well-defined map, still depending on the base point $p_\circ$, called the Abel–Jacobi map:

$$A : C \to \mathrm{Jac}(C).$$

The Abel–Jacobi map can be extended to $\mathrm{Div}(C)$ or to $\mathrm{Div}^0(C)$. The Abel–Jacobi map $A_0 : \mathrm{Div}^0(C) \to \mathrm{Jac}(C)$ on divisors of degree 0 is independent of the chosen base point $p_\circ$.

Theorem 2.4.4. 1. *(Abel's Theorem) A divisor D of degree 0 on C is the divisor of a rational function on C if and only if $A_0(D)$ is trivial in $\mathrm{Jac}(C)$.*

2. *(Jacobi's Theorem) The map $A_0 : \mathrm{Div}_0(C) \to \mathrm{Jac}(C)$ is surjective.*

3. *Thus, there is an isomorphism:*

$$\mathrm{Pic}^0(C) \cong \mathrm{Jac}(C).$$

In light of Theorem 2.4.4, we will identify $\mathrm{Pic}^0(C)$ and $\mathrm{Jac}(C)$ without comment in later chapters.

2.4.4 Variations on the Abel–Jacobi map

Let $\mathrm{Sym}_g(C)$ be C^g/S_g where S_g denotes the symmetric group on g letters. The objects in $\mathrm{Sym}_g(C)$ are unordered sets $\{x_1, \ldots, x_g\}$ of g points of C. Define a map

$$\psi_g : \mathrm{Sym}_g(C) \to \mathrm{Pic}^0(C),$$

taking $\{x_1, \ldots, x_g\}$ to the class of $\sum_{i=1}^{g}[x_i] - g[p_\circ]$.

These facts follow from the Riemann–Roch theorem:

If D is any divisor of degree 0 on C, then there exist points $x_1, \ldots, x_g$ on C such that D is equivalent to $[x_1] + \cdots [x_g] - g[P_0]$. As a result, ψ_g is surjective.

It also follows from the Riemann–Roch Theorem that ψ_g is generically injective.

Similarly, there is a map $\alpha : C \to \mathrm{Pic}^0(C)$, which takes x to the class of $[x] - [p_\circ]$, which is equivalent to the Abel–Jacobi map.

Theorem 2.4.5. *The map $\alpha : C \to \mathrm{Pic}^0(C)$ is an embedding.*

2.4.5 Torelli's Theorem

Every smooth curve X over k is uniquely determined by its Jacobian.

Theorem 2.4.6. *(Torelli's Theorem) Suppose C and C' are two smooth projective curves of genus g. If $\mathrm{Jac}(C)$ and $\mathrm{Jac}(C')$ are isomorphic as principally polarized abelian varieties, then C and C' are isomorphic as curves.*

2.4.6 The Torelli morphism

The Torelli morphism $\tau_g : \mathcal{M}_g \to \mathcal{A}_g$ takes a curve X to its Jacobian J_X.

Theorem 2.4.7. *(Torelli's Theorem, see [MFK94, Section 7.4]) If k is an algebraically closed field, then the Torelli map $T : \mathcal{M}_g(k) \to \mathcal{A}_g(k)$ is injective.*

Definition 2.4.8. The *open Torelli locus* $\mathcal{T}_g^\circ$ is the image of $\mathcal{M}_g$ under τ. it is the locus of all principally polarized abelian varieties of dimension g that are Jacobians of smooth curves.

2.5 Related results

2.5.1 Compactifications

A (marked) nodal curve is *stable* if its automorphism group is finite.

We say that C has *compact type* if each irreducible component of C is smooth and if the dual graph of C is a tree. Curves which are not of compact type correspond to points of a component Δ_0 (defined in Section 5.2.1) of the boundary $\partial\bar{\mathcal{M}}_g$.

In Section 5.2.1, we define the Picard group (or Jacobian) of a singular stable curve. The Picard variety $\mathrm{Pic}^0(C)$ is an abelian variety if and only if C has compact type. If not, then $\mathrm{Pic}^0(C)$ is a semi-abelian variety.

Let $\tilde{\mathcal{A}}_g$ be a toroidal compactification of $\mathcal{A}_g$.

Let $\bar{\mathcal{M}}_g$ denote the Deligne-Mumford compactification of $\mathcal{M}_g$. Its points represent stable curves of genus g. Let $\mathcal{M}_g^{ct}$ denote the subspace whose points represent curves of compact type.

The Torelli morphism extends to a morphism $\tau : \bar{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g$. It is no longer injective, as seen in Fact 2.5.1.

Fact 2.5.1. *Torelli's Theorem 2.4.6 is false for stable curves.*

Example 2.5.2. Consider a curve C of genus 3 that has two components: C_1, an elliptic curve; and C_2, a curve of genus 2. These are identified (clutched together) at the identity on C_1 and a point $P \in C_2$. There is a one-parameter family of such curves, as the point $P \in C_2$ varies. However, $\mathrm{Jac}(C)$ is isomorphic to $\mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2)$, and this does not depend on the choice of P.

The *closed Torelli locus* $\mathcal{T}_g$ is the image of $\mathcal{M}_g^{ct}$ under τ.

2.5.2 A stacky perspective

To summarize, we defined several moduli spaces of abelian varieties and curves. Technically, these are categories, each of which is fibered in groupoids over the category of k-schemes in its étale topology:

$\mathcal{A}_g$ principally polarized abelian schemes of dimension g;

$\tilde{\mathcal{A}}_g$ principally polarized semi-abelian schemes of dimension g;

$\mathcal{M}_g$ smooth connected proper relative curves of genus g;

$\bar{\mathcal{M}}_g$ stable relative curves of genus g.

For each positive integer r, there is also (see [Knu83, Def. 1.1,1.2]):

$\bar{\mathcal{M}}_{g;r}$ the moduli space of r-labeled stable relative curves $(C; P_1, \ldots, P_r)$ of genus g.

Each of the moduli spaces above is a smooth Deligne-Mumford stack. Furthermore, $\bar{\mathcal{M}}_g$ and $\bar{\mathcal{M}}_{g;r}$ are proper [Knu83, Theorem 2.7]. Likewise, $\tilde{\mathcal{A}}_g$ is proper.

For a moduli space $\mathcal{M}$ and a k-scheme T, by definition $\mathcal{M}(T) = \mathrm{Mor}_k(T, \mathcal{M})$ is the category of T-objects in $\mathcal{M}$ defined over T.

There is a tautological abelian variety $\mathcal{X}_g$ over the moduli stack $\mathcal{A}_g$. If $s \in \mathcal{A}_g(k)$, let $\mathcal{X}_{g,s}$ denote the fiber of $\mathcal{X}_g$ over s, which is the principally polarized abelian variety represented by the point $s : \mathrm{Spec}(k) \to \mathcal{A}_g$. There is a tautological curve $\mathcal{C}_g$ over the moduli stack $\mathcal{M}_g$ [DM69, Section 5]. If $s \in \mathcal{M}_g(k)$, let $\mathcal{C}_{g,s}$ denote the fiber of $\mathcal{C}_g$ over s, which is the curve represented by the point $s : \mathrm{Spec}(k) \to \mathcal{M}_g$.

2.5.3 The Schottky problem

The Schottky problem asks for a characterization of the p.p. abelian varieties that are Jacobians of curves. There is a lot of important work on this problem; for example, see Welters [Wel83, Wel84], Shiota [Shi86], Krichever [Kri06], [Kri10] and Arbarello, Krichever, & Marini [AKM06].

2.6 Open questions

Ekedahl and Serre asked the following question. They provided examples for numerous values of g up to 1297.

Question 2.6.1. *[ES93] Given $g \geq 2$, does there exist a smooth curve X of genus g such that the Jacobian J_X is isogenous to a product of g elliptic curves?*

The recent paper by Paulhus and Rojas [PR17] shows that the question has an affirmative answer for a lot of new values of g. It also includes references to other papers on this topic. At that time, the smallest genus for which the answer was not known was $g = 38$ but recently that genus was resolved using a modular curve https://beta.lmfdb.org/ModularCurve/Q/60.540.38.bk.1/ It seems that the smallest genus for which the answer is not known is $g = 59$, with the next smallest genus being $g = 66$.

Chapter 3

Arithmetic Invariants

3.1 Overview

Let k be an algebraically closed field of positive characteristic p. An elliptic curve over k can be ordinary or supersingular, depending on how many p-torsion points it has, see Sections 3.1.1 and 3.1.2. This section describes several ways to generalize the distinction between ordinary and supersingular for abelian varieties of dimension greater than 1.

Suppose X is a principally polarized abelian variety of dimension g defined over k. This section contains the definition of these arithmetic invariants: the **p-rank**, the **Newton polygon**, the **a-number**, and the **Ekedahl–Oort type**. If C is a curve of genus g, the invariants of C are defined to be that of its Jacobian.

A more complete description of the material in this section can be found in these references: [LO98], [Oor01b], or the chapter *Moduli of Abelian Varieties* by Chai and Oort.

3.1.1 Collapsing of p-torsion points modulo p

Suppose E is an elliptic curve over k. In this expository section, we show through some examples that the number of p-torsion points on E is either p or 1.

If $\ell \neq p$ is prime, then there are ℓ^2 points of order dividing ℓ on E. One of these is the point at infinity O_E. The x-coordinates of the other points are the roots of the ℓ-division polynomial of x.

Example 3.1.1. Write $E : y^2 = x^3 + ax^2 + bx + c$. Let $\ell = 3$. A point Q has order 3 if and only if $3Q = 0_E$, equivalently $2Q = -Q$, equivalently $x(2Q) = x(Q)$. Using this, we can show that Q has order 3 if and only if $x(Q)$ is a root of the 3-division polynomial:

$$d_3(x) = 3x^4 + 4ax^3 + 6bx^2 + 12cx - b^2 + 4ac.$$

If $p \neq \ell$, then $d_3(x)$ has 4 distinct roots in k and these are the x-coordinates of points of order 3 on E. For each x-coordinate, there are two choices for y, so E has 8 points of order 3. Together with O_E, this gives 9 points that are 3-torsion points.

Now suppose that $p = 3$. Note that $d_3(x) \equiv ax^3 - b^2 + ac$. This has one (triple) root if $a \not\equiv 0 \bmod 3$ and has no roots if $a \equiv 0 \bmod 3$. So the number of 3-torsion points is either 3 or 1, not 9.

Example 3.1.2. Write $E : y^2 = x^3 + bx + c$. The reduction of the 5-division polynomial modulo 5 is $2bx^{10} - b^2cx^5 + b^6 - 2b^3c^2 - c^4$. This has either 2 or zero roots, so the number of 5-torsion points is either 5 or 1.

The reduction of the 7-division polynomial modulo 7 is

$$3cx^{21} + 3b^2c^2x^{14} + (-b^7c - 2b^4c^3 + 3bc^5)x^7 - b^{12} - b^9c^2 + 3b^6c^4 - b^3c^6 + 2c^8.$$

This has either 3 or zero roots, so the number of 7-torsion points is either 7 or 1.

More generally, the reduction of the p-division polynomial modulo p has either $(p-1)/2$ or zero roots. As a result, the p-torsion points on $E : y^2 = f(x)$ collapse to either p points or 1 point modulo p. However, it is not easy to show this explicitly for larger p because the p-division polynomials become more and more complicated.

3.1.2 Supersingular elliptic curves

Suppose that E is an elliptic curve defined over a finite field $\mathbb{F}_q$ where $q = p^r$. Let $a \in \mathbb{Z}$ be such that $\#E(\mathbb{F}_q) = q + 1 - a$. The zeta function of $E/\mathbb{F}_q$ is

$$Z(E/\mathbb{F}_q, T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}.$$

The supersingular condition was studied by Deuring [Deu41]. As seen in [Sil09, Theorem V.3.1], there are many equivalent ways to define what it means for E to be supersingular. In this section, we say $E/\mathbb{F}_q$ is supersingular when $p \mid a$, see [Sil09, page 142]; otherwise E is ordinary.

If $p = 2$, then $E : y^2 + y = x^3$ is supersingular, see Lemma 4.4.1. In fact, this is an equation for the unique isomorphism class of supersingular elliptic curve over $\overline{\mathbb{F}}_2$.

By [Sil09, Example V.4.4], the elliptic curve $E : y^2 = x^3+1$ (j-invariant 0) is supersingular if and only if $p \equiv 2 \bmod 3$ and p is odd. By [Sil09, Example V.4.5], the elliptic curve $E : y^2 = x^3 + x$ (j-invariant 1728) is supersingular if and only if $p \equiv 3 \bmod 4$. When $p = 3$, this is an equation for the unique isomorphism class of supersingular elliptic curve over $\overline{\mathbb{F}}_3$.

Suppose p is odd and $E : y^2 = h(x)$, where $h(x)$ is a cubic with distinct roots. Then E is supersingular if and only if the coefficient c_{p-1} of x^{p-1} in $h(x)^{(p-1)/2}$ is zero.

As we will see in Example 4.2.8. this coefficient vanishes if and only if the Cartier operator trivializes $\frac{dx}{y} \in H^0(E, \Omega^1)$. As seen in [Sil09, Theorem V.4.1], for p odd, Igusa proved that

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

is supersingular for exactly $(p-1)/2$ choices of $\lambda \in \overline{\mathbb{F}}_p$; this shows that the number of isomorphism classes of supersingular elliptic curves is $\lfloor \frac{p}{12} \rfloor + \epsilon$ with $\epsilon = 0, 1, 1, 2$ when $p \equiv 1, 5, 7, 11 \bmod 12$ respectively.

Also, every supersingular elliptic curve which is defined over a field of characteristic p is, in fact, defined over $\mathbb{F}_{p^2}$.

3.1.3 Ordinary and supersingular elliptic curves

To begin, we revisit the case of elliptic curves and describe the distinction between ordinary and supersingular elliptic curves from several other points of view.

Let E/k be an elliptic curve and let ℓ be prime. The ℓ-torsion group scheme $E[\ell]$ of E is the kernel of the multiplication-by-ℓ morphism $[\ell] : E \to E$. Then

$$\#E[\ell](k) = \begin{cases} \ell^2 & \text{if } \ell \neq p \\ \ell & \text{if } \ell = p, E \text{ ordinary} \\ 1 & \text{if } \ell = p, E \text{ supersingular} \end{cases} .$$

In a later section, we will define the following terms and show that the following conditions are equivalent to E being ordinary: E has p points of order dividing p; the Newton polygon of E has slopes 0 and 1; or the group scheme $E[p]$ is isomorphic to $L := \mathbb{Z}/p \oplus \mu_p$.

The following conditions are equivalent to E being supersingular:

(A)' The only p-torsion point of E is the identity: $E[p](k) = \{\mathrm{id}\}$.

(B)' The Newton polygon of E is a line segment of slope 1/2.

(C)' The group scheme $E[p]$ is isomorphic to $I_{1,1}$, the unique local-local symmetric BT_1 group scheme of rank p^2.

Conditions (A)' and (B)' are equivalent by [Sil09, Theorem V.3.1 and page 142].

More information about group schemes and condition (C)' can be found in [Gor02, Appendix A, Example 3.14]. Briefly, consider the group scheme α_p which is the kernel of Frobenius on G_a. As a k-scheme, $\alpha_p \simeq \mathrm{Spec}(k[x]/x^p)$ with co-multiplication $m^*(x) = x \otimes 1 + 1 \otimes x$ and co-inverse $\mathrm{inv}^*(x) = -x$. The group scheme $I_{1,1}$ fits in a non-split exact sequence

$$0 \to \alpha_p \to I_{1,1} \to \alpha_p \to 0. \tag{3.1}$$

Let $D_{1,1}$ be the mod p Dieudonné module of $I_{1,1}$, see Example 3.2.7.

3.2 Background

Let k be an algebraically closed field of characteristic $p > 0$. Let X be a principally polarized abelian variety of dimension g defined over k.

In this section, we will define the following arithmetic invariants of X:

A. p-rank - the integer f, with $0 \leq f \leq g$, such that $\#X[p](k) = p^f$.

B. Newton polygon - the data of slopes for the p-divisible group $X[p^\infty]$.

C. Ekedahl-Oort type - the data defining the symmetric BT_1 group scheme $X[p]$.

3.2.1 The p-torsion group scheme

The multiplication-by-p morphism $[p] : X \to X$ is a finite flat morphism of degree p^{2g}. There is a canonical factorization $[p] = \mathrm{Ver} \circ F$, where $F : X \to X^{(p)}$ denotes the relative Frobenius morphism and $\mathrm{Ver} : X^{(p)} \to X$ is the Verschiebung morphism. The morphism F comes from the p-power map on the structure sheaf; it is purely inseparable of degree p^g. Also V is the dual of $F_{X^{\mathrm{dual}}}$.

The *p-torsion group scheme* of X is

$$X[p] = \mathrm{Ker}[p].$$

In fact, $X[p]$ is a symmetric BT_1 group scheme as defined in [Oor01b, 2.1, Definition 9.2]. It has rank p^{2g}. It is killed by $[p]$, with $\mathrm{Ker}(F) = \mathrm{Im}(\mathrm{Ver})$ and $\mathrm{Ker}(\mathrm{Ver}) = \mathrm{Im}(F)$.

The principal polarization on X induces a principal quasipolarization (pqp) on $X[p]$, i.e., an anti-symmetric isomorphism $\psi : X[p] \to X[p]^D$, where D denotes the Cartier dual. (This definition needs to be modified slightly if $p = 2$.) Thus, $X[p]$ is a symmetric BT_1 group scheme together with a principal quasipolarization.

We will define the return to this topic in Section 3.2.7 when defining the Ekedahl–Oort type.

3.2.2 The p-rank and a-number

The *p-rank* of X is

$$f = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, X),$$

where μ_p is the kernel of Frobenius on G_m. The advantage of this definition is that it is also valid for semi-abelian varieties.

When X is an abelian variety, then the p-rank determines the number of p-torsion points on X; namely p^f is the cardinality of $X[p](k)$. The reason is that the multiplicity of the group schemes $\mathbb{Z}/p$ and μ_p in $X[p]$ is the same because of the symmetry induced by the polarization.

The *a-number* of X is

$$a = \dim_k \mathrm{Hom}(\alpha_p, X),$$

where α_p is the kernel of Frobenius on G_a. It is known that $0 \leq f \leq g$ and $1 \leq a + f \leq g$.

Definition 3.2.1. The abelian variety X is *ordinary* if $f = g$; equivalently, X is ordinary if $a > 0$.

Since μ_p and α_p are both simple group schemes, the p-rank and a-number are additive;

$$f(X_1 \times X_2) = f(X_1) + f(X_2) \text{ and } a(X_1 \times X_2) = a(X_1) + a(X_2). \tag{3.2}$$

The p-rank and a-number can also be defined for a p-torsion group scheme, p-divisible group, or Dieudonné module.

3.2.3 The p-divisible group

For each $n \in \mathrm{N}$, consider the multiplication-by-p^n morphism $[p^n] : X \to X$ and its kernel $X[p^n]$. The p-divisible group of X is $X[p^\infty] = \varinjlim X[p^n]$.

For each pair (c, d) of non-negative relatively prime integers, fix a p-divisible group $G_{c,d}$ of codimension c, dimension d, and thus height $c+d$. By the Dieudonné-Manin classification [Man63], there is an isogeny of p-divisible groups

$$X[p^\infty] \sim \oplus_{\lambda = \frac{d}{c+d}} G_{c,d}^{m_\lambda}, \tag{3.3}$$

where (c, d) ranges over pairs of non-negative relatively prime integers.

Definition 3.2.2. A principally polarized abelian variety X is *supersingular* if $\lambda = 1/2$ is the only slope of its p-divisible group $X[p^\infty]$.

Letting $G_{1,1}$ denote the p-divisible group of dimension 1 and height 2, then X is supersingular if and only $X[p^\infty] \sim G_{1,1}^g$ [LO98, Section 1.4].

There are several other ways to characterize the supersingular property for an abelian variety X defined over a finite field $\mathbb{F}_q$. Write $q = p^n$. Consider the characteristic polynomial $P(X/\mathbb{F}_q, T)$ of Frobenius on X (or its ℓ-adic Tate module, for $\ell \neq p$). It is a monic polynomial of degree $2g$ with integer coefficients. Then $P(X/\mathbb{F}_q, T) = \prod_{i=1}^{2g}(T - \alpha_i)$ where $|\alpha_i| = \sqrt{q}$. These facts imply that there are integers $a_1, \ldots, a_g$ such that

$$P(X/\mathbb{F}_q, T) = T^{2g} + a_1 T^{2g-1} + \cdots + a_g T^g + q a_{g-1} T^{g-1} + \cdots + q^g. \tag{3.4}$$

Theorem 3.2.3. *A principally polarized abelian variety $X/\mathbb{F}_q$ is supersingular if and only if:*

1. *the integer a_r is divisible by $p^{\lceil rn/2 \rceil}$ for $1 \leq r \leq g$ (Manin) [Oor74, page 116];*
2. $\mathrm{End}_{\overline{\mathbb{F}}_q}(X) \otimes \mathbb{Q} \simeq \mathrm{Mat}_g(D_p)$, *where D_p is the quaternion algebra ramified only over p and ∞ [Tat66, Theorem 2d];*
3. *X is geometrically isogenous to E^g for some supersingular elliptic curve $E/\overline{\mathbb{F}}_p$ [Oor74, Theorem 4.2], which relies on [Tat66, Theorem 2d].*

3.2.4 The Newton polygon

The Newton polygon is an invariant of $X[p^\infty]$, and thus an invariant of X. Recall (3.3). The *Newton polygon* $\nu(X)$ is the multi-set of values of λ, which are called the *slopes*. It is determined by the multiplicities m_λ.

Lemma 3.2.4. *The p-rank of X is the multiplicity of the slope* 0 *in* $\nu(X)$.

For $\lambda \in \mathbb{Q} \cap [0, 1]$, the multiplicity m_λ is the multiplicity of λ in the multi-set; if $c, d \in \mathbb{N}$ are relatively prime integers such that $\lambda = c/(c + d)$, then $(c + d)$ divides m_λ. The Newton polygon is *symmetric* if $m_\lambda = m_{1-\lambda}$ for every $\lambda \in \mathbb{Q} \cap [0, 1]$. The Newton polygon is typically drawn as a lower convex polygon, with slopes equal to the values of λ occurring with multiplicity m_λ. The Newton polygon of a g-dimensional abelian variety X is symmetric and, when drawn as a polygon, it has endpoints $(0, 0)$ and $(2g, g)$ and integral break points.

There is a partial ordering on Newton polygons of the same height $2g$: one Newton polygon is smaller than a second if the lower convex hull of the first is never below the second. We write $\nu_1 \leq \nu_2$ if ν_1, ν_2 share the same endpoints and ν_1 lies on or above ν_2. This defines a partial ordering on Newton polygons for abelian varieties of dimension g. In this partial ordering, the ordinary Newton polygon is maximal and the supersingular Newton polygon is minimal.

If X_1 and X_2 are isogenous, then they have the same Newton polygon.

3.2.5 The Newton polygon, version 2

Suppose X is defined over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Then there exists a finite subfield $\mathbb{F}_0 \subset \mathbb{F}$ such that X is isomorphic to the base change to $\mathbb{F}$ of an abelian scheme X_0 over $\mathbb{F}_0$. Let $W(\mathbb{F}_0)$ denote the Witt vector ring of $\mathbb{F}_0$. Consider the action of Frobenius φ on the crystalline cohomology group $H^1_{\mathrm{cris}}(X_0/W(\mathbb{F}_0))$. There exists an integer n, for example $n = [\mathbb{F}_0 : \mathbb{F}_p]$, such that the composition of n Frobenius actions φ^n is a linear map on $H^1_{\mathrm{cris}}(X_0/W(\mathbb{F}_0))$.

In this situation, the *Newton polygon* $\nu(X)$ of X is the multi-set of rational numbers λ such that $n\lambda$ are the valuations at p of the eigenvalues of φ^n. Note that the Newton polygon is independent of the choice of X_0, $\mathbb{F}_0$, and n.

Notation 3.2.5. We use $\oplus$ to denote the union of multi-sets. For any multi-set ν, and $n \in \mathbb{N}$, we write ν^n for the union of n copies of ν.

Let ord denote the Newton polygon $\{0, 1\}$ and ss denote the Newton polygon $\{1/2, 1/2\}$. Let σ_g denote the supersingular Newton polygon of height $2g$. Thus an ordinary (resp. supersingular) abelian variety of dimension g has Newton polygon ord^g (resp. $\sigma_g = ss^g$).

For $s, t \in \mathbb{N}$, with $s \leq t/2$ and $\gcd(s, t) = 1$, let $(s/t, (t-s)/t)$ denote the Newton polygon with slopes s/t and $(t-s)/t$, each with multiplicity t.

3.2.6 Dieudonné modules

The p-divisible group $X[p^\infty]$ and the p-torsion group scheme $X[p]$ can be described using covariant Dieudonné theory, see e.g., [Oor01b, 15.3]. Differences between the covariant and contravariant theory do not cause a problem in this manuscript since all objects we consider are principally quasipolarized and thus symmetric.

Briefly, let σ denote the Frobenius automorphism of k and its lift to the Witt vectors $W(k)$. Consider the semi-linear operators F and V on $X[p]$ where F is σ-linear and V is σ^{-1}-linear. Let $\tilde{\mathbb{E}} = \tilde{\mathbb{E}}(k) = W(k)[F, V]$ denote the non-commutative ring generated by F and V with relations

$$FV = VF = p, \; F\tau = \tau^\sigma F, \; \tau V = V\tau^\sigma, \tag{3.5}$$

for all $\tau \in W(k)$.

There is an equivalence of categories $\mathbb{D}_*$ between p-divisible groups over k and $\tilde{\mathbb{E}}$-modules which are free of finite rank over $W(k)$. For example, the Dieudonné module $D_\lambda := \mathbb{D}_*(G_{c,d})$ is a free $W(k)$-module of rank $c + d$. Over $\mathrm{Frac}\, W(k)$, there is a basis $x_1, \ldots, x_{c+d}$ for D_λ such that $F^d x_i = p^c x_i$.

We now consider Dieudonné modules modulo p. Let $\mathbb{E} = \tilde{\mathbb{E}} \otimes_{W(k)} k$ be the reduction of the Cartier ring modulo p; it is a non-commutative ring $k[F, V]$ subject to the same constraints as (4.1), except that $FV = VF = 0$ in $\mathbb{E}$. Again, there is an equivalence of categories $\mathbb{D}_*$ between finite commutative group schemes I (of rank $2g$) annihilated by p and $\mathbb{E}$-modules of finite dimension $(2g)$ over k.

For elements $w_1, \ldots, w_r \in \mathbb{E}$, let $\mathbb{E}(w_1, \ldots, w_r)$ denote the left ideal $\sum_{i=1}^r \mathbb{E} w_i$ of $\mathbb{E}$ generated by $\{w_i \mid 1 \leq i \leq r\}$.

The mod p Dieudonné module of X is an $\mathbb{E}$-module of finite dimension $(2g)$.

Example 3.2.6. If E is an ordinary elliptic curve, then $E[p] \cong \mu_p \oplus \mathbb{Z}/p\mathbb{Z}$ and the mod p Dieudonné module for E is isomorphic to $L := \mathbb{E}/\mathbb{E}(F, V-1) \oplus \mathbb{E}/\mathbb{E}(V, F-1)$.

Example 3.2.7. *The group scheme $I_{1,1}$.* There is a unique symmetric BT_1 group scheme of rank p^2 and p-rank 0, which we denote $I_{1,1}$. It is a non-split extension of α_p by α_p as in (3.1). The mod p Dieudonné module of $I_{1,1}$ is $D_{1,1} := \mathbb{D}_*(I_{1,1})$. Then $D_{1,1} \simeq \mathbb{E}/\mathbb{E}(F+V)$.

If E is a supersingular elliptic curve, then $E[p] \cong I_{1,1}$ and the mod p Dieudonné module for E is $D_{1,1}$.

Remark 3.2.8. If $M = \mathbb{D}_*(I)$ is the Dieudonné module over k of I, then a principal quasipolarization $\psi : I \to I^D$ induces a a nondegenerate symplectic form $\langle \cdot, \cdot \rangle : M \times M \to k$ on the underlying k-vector space of M, subject to the additional constraint that, for all x and y in M,

$$\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma.$$

3.2.7 The Ekedahl-Oort type

The p-torsion $X[p]$ of X is a symmetric BT_1-group scheme (of rank $2g$) annihilated by p.

Isomorphism classes of pqp BT_1 group schemes over k have been completely classified in terms of Ekedahl-Oort types [Oor01b, Theorem 9.4 & 12.3], see Section 3.2.7. This builds on work of Kraft [Kra] (unpublished, which did not include polarizations) and of Moonen [Moo01] (for $p \geq 3$). (When $p = 2$, there are complications with the polarization which are resolved in [Oor01b, 9.2, 9.5, 12.2].)

As in [Oor01b, Sections 5 & 9], the isomorphism type of a symmetric BT_1 group scheme I over k can be encapsulated into combinatorial data. If I is symmetric with rank p^{2g}, then there is a *final filtration* $N_1 \subset N_2 \subset \cdots \subset N_{2g}$ of $\mathbb{D}_*(I)$ as a k-vector space which is stable under the action of V and F^{-1} such that $i = \dim(N_i)$ [Oor01b, 5.4].

The *Ekedahl-Oort type* of I is

$$\nu = [\nu_1, \ldots, \nu_g], \text{ where } \nu_i = \dim(V(N_i)).$$

Lemma 3.2.9. *The p-rank is* $\max\{i \mid \nu_i = i\}$ *and the a-number equals $g - \nu_g$.*

There is a restriction $\nu_i \leq \nu_{i+1} \leq \nu_i + 1$ on the Ekedahl-Oort type. There are 2^g Ekedahl-Oort types of length g since all sequences satisfying this restriction occur. By [Oor01b, 9.4, 12.3], there are bijections between (i) Ekedahl-Oort types of length g; (ii) pqp BT_1 group schemes over k of rank p^{2g}; and (iii) pqp Dieudonné modules of dimension $2g$ over k.

By [EvdG09], the Ekedahl-Oort type can also be described by its Young type μ. Given ν, for $1 \leq j \leq g$, consider the strictly decreasing sequence

$$\mu_j = \#\{i \mid 1 \leq i \leq g,\ i - \nu_i \geq j\}.$$

There is a Young diagram with μ_j squares in the jth row. (Unlike in combinatorics, we draw the Young diagrams to look like a staircase, ascending to the right.) The *Young type* is $\mu = \{\mu_1, \mu_2, ...\}$, where one eliminates all μ_j which are 0.

Lemma 3.2.10. *The p-rank is $g - \mu_1$ and the a-number is $a = \max\{j \mid \mu_j \neq 0\}$.*

The Ekedahl-Oort type places restrictions on the Newton polygon and vice-versa, see [Har07a, Har10].

Example 3.2.11. Let $r \in \mathrm{N}$. There is a unique symmetric BT_1 group scheme of rank p^{2r} with p-rank 0 and a-number 1, which we denote $I_{r,1}$. The Dieudonné module of $I_{r,1}$ has the property that $\mathbb{D}_*(I_{r,1}) \simeq \mathbb{E}/\mathbb{E}(F^r + V^r)$. For $I_{r,1}$, the Ekedahl-Oort type is $[0, 1, 2, \dots, r-1]$ and the Young type is $\{r\}$.

3.3 Main theorems

3.3.1 The difference between p-rank 0 and supersingular

Let X be a principally polarized abelian variety of dimension g over k. Let $X[p]$ be the kernel of the multiplication-by-p morphism of A. The following conditions are all different for $g \geq 3$.

(A) ***p*-rank** 0 - The only p-torsion point of X is the identity: $A[p](k) = \{\mathrm{id}\}$.

(B) supersingular - The Newton polygon of X is a line segment of slope $1/2$.

(C) superspecial - The group scheme $X[p]$ is isomorphic to $(I_{1,1})^g$.

Proposition 3.3.1. *For conditions $(A), (B), (C)$ as defined above, there is an implication:*

$$(C) \Rightarrow (B) \Rightarrow (A), \text{ but } (A) \overset{g\geq 3}{\not\Rightarrow} (B) \overset{g \geq 2}{\not\Rightarrow} (C).$$

Proof. (Sketch)

1. For the implication $(C) \Rightarrow (B)$: if the p-torsion of a p-divisible group G satisfies (C), then $F^2G \subset [p]G$. By the basic slope estimate in [Kat79, 1.4.3], the slopes of the Newton polygon are all at least $1/2$; so the slopes all equal $1/2$, because the polarization forces the Newton polygon to be symmetric. Thus X is supersingular. Alternatively, the implication $(C) \Rightarrow (B)$ follows from [Oor75, Theorem 2] and [Oor74, Theorem 4.2].

2. For the non-implication $(B) \not\Rightarrow (C)$ when $g \geq 2$: an abelian variety can be isogenous but not isomorphic to a product of supersingular elliptic curves; for example, quotients of a superspecial abelian variety by an α_p-subgroup scheme have this property when $g \geq 2$.

3. For the implication $(B) \Rightarrow (A)$: more generally, the p-rank of a p-divisible group is the multiplicity of the slope 0 in the Newton polygon, so if all the slopes equal $1/2$, then the p-rank is 0; Alternatively, if X is the Jacobian of a curve defined over a finite field, then the p-rank equals the number of roots of the L-polynomial that are p-adic units, which equals the multiplicity of the slope 0 in the Newton polygon.

4. For the non-implication $(A) \not\Rightarrow (B)$ when $g \geq 3$: there exists a principally polarized abelian variety whose Newton polygon has slopes $1/g$ and $(g-1)/g$; it has p-rank 0 but is not supersingular when $g \geq 3$.

□

3.4 Related results

3.4.1 Examples for low dimension

In this section, we include data for $g = 2, 3, 4$. See Example 3.2.11 for the definition of $I_{r,1}$. The tables in this section previously appeared in [Pri08].

The case $g = 2$

The following table shows the 4 symmetric BT_1 group schemes that occur for principally polarized abelian surfaces. They are listed by name, together with their codimension in $\mathcal{A}_2$, p-rank f, a-number a, Ekedahl-Oort type ν, Young type μ, Dieudonné module, and Newton polygon slopes. Recall that $L = \mathbb{Z}/p \oplus \mu_p$.

Name	cod	f	a	ν	μ	Dieudonné module	Newton polygon
L^2	0	2	0	$[1,2]$	$\emptyset$	$D(L)^2$	$0,0,1,1$
$L \oplus I_{1,1}$	1	1	1	$[1,1]$	$\{1\}$	$D(L) \oplus D_{1,1}$	$0, \frac{1}{2}, \frac{1}{2}, 1$
$I_{2,1}$	2	0	1	$[0,1]$	$\{2\}$	$\mathbb{E}/\mathbb{E}(F^2+V^2)$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
$(I_{1,1})^2$	3	0	2	$[0,0]$	$\{2,1\}$	$(D_{1,1})^2$	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

The last two rows contain all the supersingular objects.

The case $g = 3$

The following table shows the 8 symmetric BT_1 group schemes that occur for principally polarized abelian threefolds.

Name	cod	f	a	ν	μ	Dieudonné module
L^3	0	3	0	$[1,2,3]$	$\emptyset$	$D(L)^3$
$L^2 \oplus I_{1,1}$	1	2	1	$[1,2,2]$	$\{1\}$	$D(L)^2 \oplus D_{1,1}$
$L \oplus I_{2,1}$	2	1	1	$[1,1,2]$	$\{2\}$	$D(L) \oplus \mathbb{E}/\mathbb{E}(F^2+V^2)$
$L \oplus (I_{1,1})^2$	3	1	2	$[1,1,1]$	$\{2,1\}$	$D(L) \oplus (D_{1,1})^2$
$I_{3,1}$	3	0	1	$[0,1,2]$	$\{3\}$	$\mathbb{E}/\mathbb{E}(F^3+V^3)$
$I_{3,2}$	4	0	2	$[0,1,1]$	$\{3,1\}$	$\mathbb{E}/\mathbb{E}(F^2+V) \oplus \mathbb{E}/\mathbb{E}(V^2+F)$
$I_{1,1} \oplus I_{2,1}$	5	0	2	$[0,0,1]$	$\{3,2\}$	$D_{1,1} \oplus \mathbb{E}/\mathbb{E}(F^2+V^2)$
$(I_{1,1})^3$	6	0	3	$[0,0,0]$	$\{3,2,1\}$	$(D_{1,1})^3$

The objects in the last two rows are always supersingular but the situation for $I_{3,1}$ and $I_{3,2}$ is more subtle. By [Oor91b, Theorem 5.12], if $A[p] \simeq I_{3,1}$, then the p-divisible group is usually isogenous to $G_{1,2} \oplus G_{2,1}$ (slopes $1/3, 2/3$) but it can also be isogenous to $G_{1,1}^3$ (supersingular). This shows that the Ekedahl-Oort stratification does not refine the Newton polygon stratification for $g \geq 3$.

3.5 Open questions

The motivation for this question will be clarified later.

Question 3.5.1. *For $5 \leq g \leq 10$, determine the Newton polygons (resp. Ekedahl–Oort types) having p-rank 0 with this property: in the partial ordering of Newton polygons (resp. Ekedahl–Oort types), the distance to the ordinary type is at most $2g-2$.*

Chapter 4

Existence of curves with given invariants

4.1 Overview

Suppose C is a smooth projective curve of genus g defined over an algebraically closed field k of characteristic p. The arithmetic invariants of C are defined to be those of its Jacobian. This chapter contains some existence results for smooth curves with certain Newton polygons or Ekedahl–Oort types. More general results about the p-rank are contained in Section 6.3.3.

Here is the motivating question.

Question 4.1.1. *If p is prime and $g \geq 2$, which p-ranks, Newton polygons, a-numbers, and Ekedahl-Oort types occur for the Jacobians of smooth curves $C/\overline{\mathbb{F}}_p$ of genus g? In particular, does there exist a smooth curve $C/\overline{\mathbb{F}}_p$ of genus g whose Jacobian (A) has p-rank 0; (B) is supersingular; or (C) is superspecial?*

In Question 4.1.1, the answer to part (A) is yes for all g and p, see Theorem 6.3.3; as seen in this section, the answer to part (B) is sometimes yes, but most often is not known; the answer to part (C) most often is not known, but is sometimes no when p is small relative to g, see Theorem 4.4.2.

In this chapter, we survey some of the results and techniques on this topic. In particular, we focus on the techniques that use cohomological calculations or decomposition of the Jacobian.

4.2 Background

4.2.1 The Newton polygon of a curve

In Sections 3.2.4 and 3.2.5, we defined the Newton polygon of an abelian variety. Here is another definition that applies for a curve over a finite field $\mathbb{F}_q$ of characteristic p. Let $C/\mathbb{F}_q$ be a smooth projective curve of genus g and let $\mathrm{Jac}(C)$ denote its Jacobian.

Definition 4.2.1. For an integer $s \geq 1$, let $N_s = \#C(\mathbb{F}_{q^s})$ be the number of points of C defined over $\mathbb{F}_{q^s}$. The zeta function of $C/\mathbb{F}_q$ is

$$Z(C/\mathbb{F}_q, T) = \exp(\sum_{s=1}^{\infty} \frac{N_s T^s}{s}).$$

Here is the famous theorem of Weil.

Theorem 4.2.2. *(Weil conjectures for curves [Wei48a, §IV, 22], [Wei48b, §IX, 69]) There is a polynomial $L(C/\mathbb{F}_q, T) \in \mathbb{Z}[T]$ of degree $2g$ such that*

$$Z(C/\mathbb{F}_q, T) = \frac{L(C/\mathbb{F}_q, T)}{(1-T)(1-qT)}.$$

Furthermore,

$$L(C/\mathbb{F}_q, T) = \prod_{i=1}^{2g} (1 - \alpha_i T),$$

where the reciprocal roots α_i of $L(C/\mathbb{F}_q, T)$ have the property that $|\alpha_i| = \sqrt{q}$.

So the roots of $L(C/\mathbb{F}_q, T)$ all have archimedean absolute value $1/\sqrt{q}$ in $\mathbb{C}$.

Lemma 4.2.3. *The characteristic polynomial of the Frobenius endomorphism of* $\mathrm{Jac}(C)$ *is* $P(\mathrm{Jac}(C)/\mathbb{F}_q, T) = T^{2g} L(C/\mathbb{F}_q, T^{-1})$.

The Newton polygon keeps track of the p-adic valuations of the roots or, equivalently, of the coefficients of $L(C/\mathbb{F}_q, T)$. Let v_i be the p-adic valuation of the coefficient of T^i in $L(C/\mathbb{F}_q, T)$. Let v_i/r be its normalization for the extension $\mathbb{F}_q/\mathbb{F}_p$, where $q = p^r$. The Newton polygon is the lower convex hull of the points $(i, v_i/r)$ for $0 \leq i \leq 2g$. The Newton polygons of $C/\mathbb{F}_q$ and $\mathrm{Jac}(C)$ are the same.

The Newton polygon consists of finitely many line segments, which break at points with integer coefficients, starting at $(0,0)$ and ending at $(2g, g)$. If the slope λ appears with multiplicity m, then so does the slope $1 - \lambda$.

Definition 4.2.4. The curve $C/\mathbb{F}_q$ is *supersingular* if the Newton polygon of $L(C/\mathbb{F}_q, T)$ is a line segment of slope $1/2$.

There are several ways to characterize the supersingular property for curves, in addition to those already described in Lemma 3.2.3.

Lemma 4.2.5. *Consider a curve $C/\mathbb{F}_q$ of genus g. The following properties are equivalent:*

1. *C is supersingular;*
2. *the normalized Weil numbers $\alpha_i/\sqrt{q}$ are all roots of unity [Man63, Theorem 4.1];*
3. *the curve C is minimal (meaning that it satisfies the lower bound in the Hasse-Weil bound for the number of points) over $\mathbb{F}_{q^s}$ for some $s \geq 1$.*

4.2.2 Computing the zeta function

Many people worked on finding fast algorithms to compute the zeta function of a curve over a finite field. There is not space to give a complete description of the literature in this area. Here are a few highlights:

In 1985, Schoof published a deterministic polynomial time algorithm for counting points on elliptic curves [Sch85].

In 2001, Kedlaya published an algorithm to compute the zeta function of a hyperelliptic curve [Ked01]. For a hyperelliptic curve of genus g over $\mathbb{F}_{p^n}$, this algorithm is polynomial in g and n. The strategy is to compute a p-adic approximation of Frobenius in the Monsky–Washnitzer cohomology. In [Har07b], Harvey made some improvements to this algorithm for large primes.

4.2.3 The Hasse–Witt and the Cartier–Manin matrices

Fix a basis for $H^0(C, \Omega^1)$. From Serre duality, this fixes a basis for the dual space $H^1(C, \mathcal{O})$. The Hasse–Witt matrix is the matrix for the action of Frobenius F on $H^1(C, \mathcal{O})$ with respect to that basis. The Cartier–Manin matrix is the matrix for the action of Vershiebung V on $H^0(C, \Omega^1)$ with respect to that basis.

By [Car57], [Man63], the matrix for V on $H^0(C, \Omega^1)$ is the same as the Cartier–Manin matrix which is the matrix for the (unmodified) Cartier operator. The (modified) Cartier operator C is the semi-linear map $C : H^0(C, \Omega^1) \to H^0(C, \Omega^1)$ satisfying these rules:

(i) $C(\omega_1 + \omega_2) = C(\omega_1) + C(\omega_2)$;

(ii) $C(f^p \omega) = f C(\omega)$; and

(iii) $C(f^{n-1} df) = \begin{cases} df & \text{if } n = p, \\ 0 & \text{if } 1 \leq n < p. \end{cases}$

Lemma 4.2.6. *The p-rank of C is the stable rank of the Cartier operator. The a-number of C is the corank of the Cartier operator.*

The p-rank can be computed as the rank of the product of twists of $\tilde{M}$ (or M) but this needs to be done very carefully as described in Remark 4.2.9.

Suppose $\beta = \{\omega_1, \ldots, \omega_g\}$ is a basis for $H^0(C, \Omega^1)$. For each ω_j, let $m_{i,j} \in k$ be such that $C(\omega_j) = \sum_{i=1}^g m_{i,j} \omega_i$. The $g \times g$-matrix $M = (m_{i,j})$ is the (modified) Cartier–Manin matrix and it gives the action of the (modified) Cartier operator. The Cartier–Manin matrix is $\tilde{M} := M^{(p)}$, where each entry is raised to the pth power.

Example 4.2.7. A formula for the Cartier operator on plane curves is given in [SV87].

Example 4.2.8. Let p be odd. Let C be a hyperelliptic curve with equation $y^2 = h(x)$. Consider the basis $\{dx/y, \ldots, x^{g-1}dx/y\}$ of $H^0(C, \Omega^1)$. By [Yui78], see also [AH19, Section 3.1], with respect to this basis, the entry $m_{i,j}$ of M is given by the coefficient of x^{pi-j} in $f(x)^{(p-1)/2}$. This is because

$$C(x^j \frac{dx}{y}) = C(x^j \frac{y^{p-1} dx}{y^p}) = \frac{1}{y} C(x^j h(x)^{(p-1)/2} dx) = \sum_{i=1}^{g} (c_{ip-j})^{1/p} \frac{dx}{y}.$$

Remark 4.2.9. Warning: if C is defined over a field field other than $\mathbb{F}_p$, it's important to be extremely careful when using Lemma 4.2.6. There are numerous mistakes in the literature about this, which were corrected in [AH19]. Because of the semi-linear property, when iterating $\tilde{M}$, the coefficients of the matrix need to be modified by pth powers. The p-rank is the rank of $\tilde{M}\tilde{M}^{(1/p)} \cdots \tilde{M}^{(p^{g-1})}$, which is the same as the rank of $\tilde{M}^{(p^{g-1})} \cdots \tilde{M}^{(p)}\tilde{M}$. This may not be the same as the rank of $\tilde{M}\tilde{M}^{(p)} \cdots \tilde{M}^{(p^{g-1})}$. The ambiguity of acting on the left or the right caused several mistakes in the literature. We refer to [AH19] for a careful analysis of this.

Example 4.2.10. In [IKO86], Ibukiyama, Katsura, and Oort count the number of superspecial curves of genus 2 in terms of p, together with the sizes of their automorphism groups. The strategy is to compute the Cartier–Manin matrix. They use Igusa's description of (families of) curves of genus 2 having extra automorphisms. For example, the curve $y^2 = (x^3 - 1)(x^3 - t)$ has an action of S_3, while the curve $y^2 = x(x^2 - 1)(x^2 - t)$ has an action of D_4. For these two families, the Cartier–Manin matrix is either invertible or is the zero matrix. In the latter case, the curve is superspecial, and thus supersingular. Using Igusa's approach with hypergeometric differential equations, they count the number of values of t for which the curve is superspecial.

Example 4.2.11. In [Mil72], Miller proved that there exists an ordinary curve of genus g over $\bar{\mathbb{F}}_p$ for all primes p and $g \geq 2$. Specifically: he proved that $y^2 = x^{2g+1} + tx^{g+1} + x$ is ordinary for a generic t if $p \nmid g$; and $y^2 = x^{2g+2} + tx^{g+1} + 1$ is ordinary for a generic t if $p \mid g$. The strategy is to find a basis for $H^0(C, \Omega^1)$ for which the Cartier–Manin matrix is a permutation matrix. The result follows by showing that the determinant is a non-zero polynomial in t.

4.2.4 The de Rham cohomology

The Ekedahl–Oort type of a curve over k can be computed from its de Rham cohomology. If C is a curve of genus g over k, then the de Rham cohomology group $H^1_{\mathrm{dR}}(C)$ is a vector space of dimension $2g$, with semi-linear operators F and V.

Recall from Section 3.2.6 that $\mathbb{E} = \mathbb{E}(k) = k[F, V]$ is the non-commutative ring generated by semilinear operators F and V with relations

$$FV = VF = 0, \ F\tau = \tau^\sigma F, \ \tau V = V\tau^\sigma, \tag{4.1}$$

for all $\tau \in k$.

Oda proved that there is an isomorphism of $\mathbb{E}$-modules between the *contravariant* Dieudonné module over k of $J_C[p]$ and $H^1_{\mathrm{dR}}(C)$ by [Oda69, Section 5]. The canonical principal polarization on J_C induces a canonical isomorphism $\mathbb{D}_*(J_C[p]) \simeq H^1_{\mathrm{dR}}(C)$.

Example 4.2.12. Suppose p is odd and C is a hyperelliptic curve. The authors of [DH] found a basis for $H^1_{dR}(C)$ and computed the action of F and V with respect to that basis.

4.3 Main theorems

4.3.1 Small genus

When g is small, there are more results about Question 4.1.1. When $g = 2$ and $g = 3$, the answer to Question 4.1.1 is known for all p, because the open Torelli locus is open and dense in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. In Section 6.3.4, we indicate how knowledge of invariants of curves of low genus can yield information about invariants of curves of higher genus.

The case $g = 2$

The open Torelli locus $\mathcal{T}_2^\circ$ is open and dense in $\mathcal{A}_2$. From this, one can check that all 3 Newton polygons and all 4 Ekedahl-Oort types occur for Jacobians of smooth curves of genus 2 over $\overline{\mathbb{F}}_p$ for all p, except for the following case: there does not exist a superspecial smooth curve of genus 2 over $\overline{\mathbb{F}}_p$ when $p = 2, 3$. This is a special case of [IKO86, Proposition 3.1], in which the authors determine the number of curves X with $\mathrm{Jac}(X)[p] \simeq (I_{1,1})^2$.

The case $g = 3$

The open Torelli locus $\mathcal{T}_3^\circ$ is open and dense in $\mathcal{A}_3$. From this, one can check that all 5 Newton polygons and all 8 Ekedahl-Oort types occur for Jacobians of smooth curves over $\overline{\mathbb{F}}_p$, except when $p = 2$ for $(I_{1,1})^3$ and $I_{1,1} \oplus I_{2,1}$.

Here are some references for the 4 bottom rows of the table, which are the p-rank 0 cases. There exists a smooth curve C of genus 3 over $\overline{\mathbb{F}}_p$ such that $\mathrm{Jac}(C)$ has the given p-torsion group scheme:

1. $I_{3,1}$, for all p by [Oor91b, Theorem 5.12(2)];
2. $I_{3,2}$, [Pri09, Lemma 4.8] for $p \geq 3$ and [EP13b, Example 5.7(3)] for $p = 2$;
3. $I_{1,1} \oplus I_{2,1}$, [Pri09, Lemma 4.8] for $p \geq 3$ (using [Oor01b, Proposition 7.3]); when $p = 2$, this group scheme does not occur as the 2-torsion of a hyperelliptic curve by [EP13b] or as the 2-torsion of a smooth plane quartic by [SV87].
4. $(I_{1,1})^3$, if and only if $p \geq 3$ by [Oor91b, Theorem 5.12(1)].

The case $g = 4$

The following result was proven by Harashita, Kudo, and Senda.

Theorem 4.3.1. *[KHS20, Corollary 1.2,1.3] For every prime p, there exists a smooth curve of genus* 4 *that is supersingular and has a-number at least* 3*.*

The construction of the proof uses curves that admit two commuting automorphisms of order 2.

Using the material in the next chapter, geometric proofs were given for the existence of curves of genus 4 with these Newton polygons:

$G_{1,3} \oplus G_{3,1}$ with slopes $1/4, 3/4$, by [AP14, Corollary 5.6]; and
$G_{1,2} \oplus G_{2,1} \oplus G_{1,1}$ with slopes $1/3, 1/2, 2/3$, by [Pri, Corollary 4.1]; and
$(G_{1,1})^4$ (supersingular), by [Pri, Corollary 1.2], see Theorem 6.3.1.

For $g = 4$, there are 16 symmetric BT_1 group schemes of rank p^8; see the table in [Pri08, Section 4.4]. There are some open questions about the Ekedahl–Oort types, specifically those with p-rank 0 and a-number at least two. For most p, for it is not known whether there are Jacobians of smooth curves of genus 4 having these Young types:

$$\{4\},\ \{4,1\},\ \{4,2\},\ \{4,3\}, \{4,2,1\},\ \{4,3,1\}, \{4,3,2\},\ \{4,3,2,1\}. \tag{4.2}$$

Here are some cases in which the answer is known:

[Zho20, Theorem 1.2] If p is odd with $p \equiv \pm 2 \bmod 5$, Zhou proved the answer is yes for the Young types $\{4,2\}$ and $\{4,3\}$.

[Zho20, Theorem 1.2] If $p \equiv 4 \bmod 5$, there exists a superspecial curve of genus 4 (Young type $\{4,3,2,1\}$).

[KHH20, Theorem 1.1], if $p < 7 < 20,000$ or $p \equiv 5 \bmod 6$, there exists a superspecial curve of genus 4.

[Drab, Corollary 6.6] If $p = 2$, Dragutinovich proved that the answer is yes for $\{4\}$, $\{4,1\}$, and $\{4,2\}$ (and the strata for these curves have the right dimension); and the answer is no for the other strata in (4.2). Similar results for $p = 3$ are in [Draa, Proposition 6.3].

4.4 Related results

4.4.1 Hermitian curves are supersingular

The Hermitian curve H_q is the curve in $\mathbb{P}^2$ defined by the affine equation $y^q + y = x^{q+1}$. Because H_q is a smooth plane curve of degree $q+1$, the genus of H_q is $g = q(q-1)/2$.

Proposition 4.4.1. *[Sti09, VI 4.4], [Han92, Proposition 3.3] The Hermitian curve H_q is maximal over $\mathbb{F}_{q^2}$. Also $L(H_q/\mathbb{F}_q, T) = (1 + qT^2)^g$ and H_q is supersingular.*

4.4.2 Non-existence of superspecial curves

This is the only non-existence result currently known for Question 4.1.1. Recall that X is superspecial if $\mathrm{Jac}(X)[p]$ is isomorphic to $(I_{1,1})^g$.

Theorem 4.4.2. *[Eke87], see also [Bak00] If $X/\overline{\mathbb{F}}_p$ is a superspecial curve of genus g, then $g \leq p(p-1)/2$.*

Theorem 4.4.2 can be stated as a non-existence result: a smooth curve of genus g defined over $\overline{\mathbb{F}}_p$ cannot be superspecial if $g > p(p-1)/2$. The Hermitian curve H_p is superspecial and its genus realizes the bound in Theorem 4.4.2.

The superspecial condition is equivalent to $a = g$ (or equivalently, $V = 0$). In [Re01], Re generalized Theorem 4.4.2, giving a bound on the genus when the a-number is large relative to g or when $V^r = 0$ for some small r.

4.4.3 Artin–Schreier curves

The situation for Artin–Schreier curves is quite different from the general case. An Artin–Schreier curve is a curve that admits a Galois cover of $\mathbb{P}^1$ that has Galois group $\mathbb{Z}/p\mathbb{Z}$. There is a lot to say about Newton polygons of Artin–Schreier curves and only a small selection of results are included here.

More generally, suppose $\pi : C_1 \to C_2$ is a Galois cover of curves with Galois group $\mathbb{Z}/p\mathbb{Z}$ such that p divides at least one of the ramification indices. In this context, the wild Riemann–Hurwitz formula [Ser68, IV] determines the genus of C_1 in terms of the genus of C_2 and the ramification jumps. Also, the Deuring–Shafarevich formula [Sub75, Theorem 4.2] determines the p-rank of C_1 in terms of the p-rank of C_2 and the ramification jumps. The relationship between the a-numbers (and the Ekedahl–Oort types) of C_1 and C_2 is more complicated, but there are some constraints; for example, see [BC20] and [CU].

There are supersingular curves of every genus in characteristic 2

Theorem 4.4.3. *[vdGvdV95, Theorem 2.1] If $p = 2$ and $g \in \mathrm{N}$, then there exists a supersingular curve Y_g of genus g defined over a finite field of characteristic 2.*

Example 4.4.4. It is possible that a Newton polygon may occur for a smooth curve in some characteristics but not in others. When $p = 2$, the Newton polygon of the curve $y^2 + y = x^{23} + x^{21} + x^{17} + x^7 + x^5$ has slopes $5/11,\ 6/11$. When $p = 2$, the Newton polygon of the curve $y^2 + y = x^{25} + x^9$ has slopes $5/12,\ 7/12$. It is not known whether these Newton polygons occur for curves in any odd characteristic. See [Oor05, Expectation 8.5.3].

There are supersingular curves of arbitrarily large genus for every odd characteristic

Theorem 4.4.5. *[vdGvdV92, Theorem 13.7], [Bla12, Corollary 3.7(ii)], [BHM⁺16, Proposition 1.8.5] If $\mathbb{F}_q$ is a finite field of characteristic p and $R(x) \in \mathbb{F}_q[x]$ is an additive polynomial of degree p^h, then $Y : y^p - y = xR(x)$ is supersingular with genus $p^h(p-1)/2$.*

We take this opportunity to fix a mistake in a published result [Pri19, Corollary 2.6].

Corollary 4.4.6. *[Karemaker/Pries] Let p be prime. Let $\delta \in \mathrm{N}$ be such that 0 and 1 are the only coefficients in the base p expansion of δ. If $g = \delta p(p-1)/2$, then there exists a supersingular curve of genus g defined over a finite field of characteristic p.*

Remark: When $p = 2$, then Corollary 4.4.6 is the same as Theorem 4.4.3 because the condition on δ is vacuous and $g = \delta$.

Proof. The condition on δ implies that, for some $t \in \mathrm{N}$,

$$\delta = \sum_{i=1}^{t} p^{s_i}(1 + p + \cdots p^{r_i}),\ \text{for some } r_i, s_i \in \mathbb{Z}^{\geq 0} \text{ such that } s_i \geq s_{i-1} + r_{i-1} + 2. \qquad (4.3)$$

Let $u_i = (s_i + 1) - \sum_{j=1}^{i-1}(r_j + 1)$ and note $u_{i+1} \geq u_i + 1$.

Choose an $\mathbb{F}_p$-linear subspace L_i of dimension $d_i := r_i + 1$ in the vector subspace of $\overline{\mathbb{F}}_p[x]$ of additive polynomials of degree p^{u_i}, with the requirement that $L_i \cap L_j = \{0\}$ if $i \neq j$. Let $\mathbb{L} = \oplus_{i=1}^t L_i$.

For $f \in \mathbb{L} - \{0\}$, let $C_f : y^p - y = xf$. By definition, C_f comes equipped with a preferred map $C_f \to \mathbb{P}^1$. If $f \in \mathbb{L} - \{0\}$ is such that it has a non-zero component in L_i, but not from L_j for $j > i$, then $g_{C_f} = p^{u_i}(p-1)/2$. By Theorem 4.4.5, $\mathrm{Jac}(C_f)$ is supersingular.

Let $\mathbb{P}(\mathbb{L})$ denote the projectivization of the $\mathbb{F}_p$-vector space L. Specifically, there is a diagonal embedding of $\mathbb{F}_p^*$ in $\mathbb{L}$. If $f_1, f_2 \in \mathbb{L} - \{0\}$, and if $f_1 = cf_2$ for some $c \in \mathbb{F}_p^*$, then the curves C_{f_1} and C_{f_2} are isomorphic over $\mathbb{F}_p$, and this isomorphism is compatible with the preferred maps to $\mathbb{P}^1$. With some abuse of notation, we write $f \in \mathbb{P}(\mathbb{L})$ to denote an equivalence class of $f \in \mathbb{L} - \{0\}$ up to scaling by constants in $\mathbb{F}_p^*$ and we write C_f for $f \in \mathbb{P}(\mathbb{L})$ to denote the curve C_f for one representative of $f \in \mathbb{L} - \{0\}$ in this equivalence class.

Let Y be the fiber product of $C_f \to \mathbb{P}^1$ for all $f \in \mathbb{P}(\mathbb{L})$. By [KR89, Theorem B], $\mathrm{Jac}(Y)$ is isogenous to $\oplus_{f \in \mathbb{P}(\mathbb{L})} \mathrm{Jac}(C_f)$. So $\mathrm{Jac}(Y)$ is supersingular. The genus of Y is $g_Y = \sum_{f \in \mathbb{P}(\mathbb{L})} g_{C_f}$.

There are $p^{d_i} - 1$ non-zero polynomials in L_i. The number of $f \in \mathbb{L}$ which have a non-zero contribution from L_i, but not from L_j for $j > i$ is $(p^{d_i} - 1) \prod_{j=1}^{i-1} p^{d_j}$. The number of equivalence classes of these f in $\mathbb{P}(\mathbb{L})$ is the quotient of this number by $p - 1$. Thus we obtain:

$$\begin{aligned} g_Y &= \sum_{i=1}^t \frac{(p^{d_i} - 1)}{p-1} \left(\prod_{j=1}^{i-1} p^{d_j}\right) p^{u_i}(p-1)/2 \\ &= \sum_{i=1}^t (p^{r_i} + \cdots + 1) p^{\sum_{j=1}^{i-1}(r_j+1)} p^{u_i - 1} p(p-1)/2 \\ &= \sum_{i=1}^t (p^{r_i} + \cdots + 1) p^{s_i} p(p-1)/2 = \delta p(p-1)/2. \end{aligned}$$

□

Ekedahl–Oort types for hyperelliptic curves when $p = 2$

Suppose $p = 2$ and C is a hyperelliptic curve. Then C is an Artin–Schreier curve, with an affine equation of the form $y^2 + y = f(x)$, for some $f(x) \in k(x)$. The combination of C being both Artin–Schreier and hyperelliptic puts a lot of constraints on its cohomology.

Theorem 4.4.7. *[EP13a] Suppose $p = 2$ and C is a hyperelliptic curve. Then $H^1_{dR}(C)$ decomposes as a module under F and V into pieces indexed by the branch points of the hyperelliptic cover. The Ekedahl–Oort type of C depends only on the ramification data and relatively few of the possible Ekedahl–Oort types occur for these curves.*

4.5 Open questions

4.5.1 Supersingular curves

Question 4.5.1. *Given a prime p and $g \in \mathbb{N}$, does there exist a smooth connected projective curve X of genus g defined over a finite field of characteristic p that is supersingular?*

When $p = 2$, the answer to Question 4.5.1 is yes for all $g \in \mathbb{N}$, see Theorem 4.4.3. For a fixed odd prime p, the answer is yes for infinitely many $g \in \mathbb{N}$, see Proposition 4.4.1, Theorem 4.4.5, and Corollary 4.4.6. In Section 4.3.1, we explain why the answer is yes for all p when $g = 1, 2, 3, 4$. The first open situation for Question 4.5.1 is when $g = 5$, for $p \not\equiv -1 \bmod 8, 11, 12, 15, 20$, and $p \not\equiv -4 \bmod 15$.

4.5.2 Counting the number of non-ordinary curves

Here is an open question that might be more tractable. The motivation will be described later.

Question 4.5.2. *Determine the rate of growth of the number of curves over $\mathbb{F}_p$ (up to geometric isomorphism) having the following types as p grows.*

1. *Non-ordinary curves of genus 4 (resp. of genus 5);*
2. *p-rank 0 curves of genus 4 (resp. of genus 5);*
3. *Supersingular curves of genus 4.*

4.5.3 Double covers of an elliptic curve

Question 4.5.3. *Let $n \geq 1$. Let E be an elliptic curve. Suppose $\phi : C \to E$ is a double cover branched at $2n$ points.*

1. *Find a basis for $H^0(C, \Omega^1)$.*
2. *Find the matrix of the Cartier operator on $H^0(C, \Omega^1)$ with respect to that basis.*
3. *Prove that the new part of $\mathrm{Jac}(C)$ is ordinary for a generic choice of $2n$ points.*
4. *Under what conditions does there exist a set of $2n$ points such that the new part of $\mathrm{Jac}(C)$ is not ordinary?*

Chapter 5

Complete subvarieties

5.1 Overview

The moduli space $\mathcal{M}_g$ is not complete, because there are families of smooth curves that specialize to singular curves. Similarly, the moduli space $\mathcal{A}_g$ is not complete, because there are families of abelian varieties that specialize to semi-abelian varieties. In this section, we describe the Deligne–Mumford compactification $\bar{\mathcal{M}}_g$ of $\mathcal{M}_g$. There are many compactifications of $\mathcal{A}_g$ [FC90]; a good reference on this topic is the survey of Hulek and Tommasi [HT18].

Specifically, in Section 5.2, we describe the boundary $\partial\mathcal{M}_g$ of $\mathcal{M}_g$. Its points represent stable singular curves of genus g. In Section 5.2.1, we describe the clutching morphisms. In Section 5.2.2, we describe the components of the boundary. In Section 5.3, we describe results about complete subvarieties of $\mathcal{M}_g$ and $\mathcal{A}_g$.

There are open questions about complete subvarieties of $\mathcal{M}_g$, meaning complete families of smooth curves. We end with an open question about the maximal dimension of a complete subvariety of $\mathcal{M}_g$.

5.2 Background: The boundary of $\mathcal{M}_g$

Recall that $\mathcal{M}_{g;r}$ is the moduli space of smooth curves of genus g together with r marked points. Let $\bar{\mathcal{M}}_{g;r}$ denote the Deligne–Mumford compactification of $\mathcal{M}_{g;r}$.

5.2.1 Clutching maps

Given two curves (with labeled points), it is possible to clutch them together to obtain a singular curve of higher genus. To set some notation, suppose g_1, g_2, r_1, r_2 are positive integers. There is a clutching map

$$\kappa_{g_1;r_1,g_2;r_2} : \bar{\mathcal{M}}_{g_1;r_1} \times \bar{\mathcal{M}}_{g_2;r_2} \longrightarrow \bar{\mathcal{M}}_{g_1+g_2;r_1+r_2-2}. \tag{5.1}$$

Suppose $s_1 \in \bar{\mathcal{M}}_{g_1;r_1}$ is the moduli point of a labeled curve $(C_1; P_1, \ldots, P_r)$, and suppose $s_2 \in \bar{\mathcal{M}}_{g_2;r_2}$ is the moduli point of a labeled curve $(C_2; Q_1, \ldots, Q_{r_2})$. Then $\kappa_{g_1;r_1,g_2;r_2}(s_1, s_2)$ is the moduli point of the labeled curve $(D; P_1, \ldots, P_{r_1-1}, Q_2, \ldots Q_{r_2})$, where the underlying curve D has components C_1 and C_2, the sections P_{r_1} and Q_1 are identified in an ordinary double point, and this nodal section is dropped from the labeling. The clutching map is a closed immersion if $g_1 \neq g_2$ or if $r_1 + r_2 \geq 3$, and is always a finite, unramified map [Knu83, Corollary 3.9].

The Jacobian of the resulting curve D is the product of the Jacobians of C_1 and C_2. Specifically, by [BLR90, Ex. 9.2.8],

$$\mathrm{Pic}^0(D) \simeq \mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2). \tag{5.2}$$

Alternatively, given a curve with two labeled points, it is possible to clutch these points together to obtain a singular curve of higher genus. To set some notation, suppose g and r are positive integers and $r \geq 2$. There is a clutching map

$$\kappa_{g;r} : \bar{\mathcal{M}}_{g;r} \longrightarrow \bar{\mathcal{M}}_{g+1;r-2}.$$

If $s \in \bar{\mathcal{M}}_{g;r}$ is the moduli point of a labeled curve $(C; P_1, \ldots, P_r)$ then $\kappa_{g;r}(s)$ is the moduli point of the labeled curve $(\tilde{C}; P_1, \ldots, P_{r-2})$ where $\tilde{C}$ is obtained by identifying the sections P_{r-1} and P_r in an ordinary double point, and these sections are dropped from the labeling. The morphism $\kappa_{g;r}$ is finite and unramified [Knu83, Corollary 3.9].

In this situation, $\mathrm{Pic}^0(\tilde{C})$ is a semi-abelian variety but not an abelian variety. By [BLR90, Ex. 9.2.8], $\mathrm{Pic}^0(\tilde{C})$ is an extension of the form

$$0 \longrightarrow W \longrightarrow \mathrm{Pic}^0(\tilde{C}) \longrightarrow \mathrm{Pic}^0(C) \longrightarrow 0\ , \tag{5.3}$$

where W is a one-dimensional torus. The toric rank of $\mathrm{Pic}^0(\tilde{C})$ is one more than the toric rank of $\mathrm{Pic}^0(C)$. The maximal projective quotient of $\tilde{C}$ is the maximal quotient which is an abelian variety; the maximal projective quotients of $\tilde{C}$ and C are isomorphic.

5.2.2 Components of the boundary

The boundary of $\mathcal{M}_g$ is $\partial\mathcal{M}_g = \bar{\mathcal{M}}_g - \mathcal{M}_g$. We will define the following components of the boundary: Δ_0, whose points represent stable curves that are not of compact type; and Δ_i for $1 \leq i \leq g/2$, whose points represent stable curves of compact type. The Jacobians of curves represented by points of Δ_0 are semi-abelian varieties, rather than abelian varieties; the Jacobians of curves represented by points of Δ_i for positive i are abelian varieties that decompose, with the product polarization.

Definition 5.2.1. Let $1 \leq i \leq g-1$ and write $g_1 = i$ and $g_2 = g - i$. Define $\Delta_i = \Delta_i[\bar{\mathcal{M}}_g]$ to be the image of $\bar{\mathcal{M}}_{i;1} \times \bar{\mathcal{M}}_{g-i;1}$ under the morphism $\kappa_{i,1;g-i,1}$, with the reduced induced structure.

The generic geometric point of Δ_i represents a curve D with two irreducible components C_1 and C_2, having genera g_1 and g_2, that intersect in an ordinary double point. Note that Δ_i and Δ_{g-i} are the same substack of $\bar{\mathcal{M}}_g$.

Definition 5.2.2. Define $\Delta_0 = \Delta_0[\bar{\mathcal{M}}_g]$ to be the image of $\bar{\mathcal{M}}_{g-1;2}$ under the morphism $\kappa_{g-1;2}$, with the reduced induced structure. Define $\mathcal{M}_g^{ct} = \bar{\mathcal{M}}_g - \Delta_0$.

The generic geometric point of Δ_0 represents a curve with one irreducible component that self-intersects in an ordinary double point. The points of $\mathcal{M}_g^{ct}$ represent curves of genus g having compact type.

Theorem 5.2.3. *[Knu83, page 190] The locus Δ_i is an irreducible divisor in $\bar{\mathcal{M}}_g$, and $\partial\mathcal{M}_g$ is the union of Δ_i for $0 \leq i \leq g/2$.*

5.3 Main theorems: Complete subvarieties

This section contains results about complete subvarieties of $\mathcal{A}_g$, $\mathcal{M}_g$, and $\bar{\mathcal{M}}_g - \Delta_0$. The proofs of these results use the structure of the Chow ring, which we do not cover here.

Theorem 5.3.1. *[Dia87a, Theorem 4] (for positive characteristic, see [Loo95b, page 412]) Suppose $g \geq 3$. If $Z \subset \mathcal{M}_g$ is complete, then $\dim(Z) \leq g - 2$.*

Theorem 5.3.2. *[Dia87b, page 80] Suppose $g \geq 3$. If $Z \subset \mathcal{M}_g^{ct}$ is complete, then $\mathrm{codim}(Z, \mathcal{M}^{ct}) \geq g$, (so $\dim(Z) \leq 2g - 3$).*

Theorem 5.3.3. *[vdG99, Corollary 1.7] Suppose $g \geq 3$. If $Z \subset \mathcal{A}_g$ is complete, then $\mathrm{codim}(Z, \mathcal{A}_g) \geq g$, (so $\dim(Z) \leq g(g-1)/2$).*

The following result of Keel and Sadun solved a conjecture of Oort [vdGO99, Conjecture 3.5].

Theorem 5.3.4. *[KS03, Corollary 1.2, 1.2.1] For $g \geq 3$, there is no complete codimension g subvariety of $\mathcal{A}_{g,\mathbb{C}}$; thus there is no complete codimension g subvariety of $\bar{\mathcal{M}}_{g,\mathbb{C}} - \Delta_0$.*

Remark 5.3.5. Both parts of Theorem 5.3.4 are false in positive characteristic: over an algebraically closed field k of characteristic $p > 0$, we will see in the next chapter that the p-rank 0 locus of $\mathcal{A}_{g,k}$ and the p-rank 0 locus of $\bar{\mathcal{M}}_{g,k} - \Delta_0$ each have codimension g and are complete.

5.4 Related results

There are many results about different compactifications of $\mathcal{A}_g$ that we do not have time to cover here. We consider $\tilde{\mathcal{A}}_g$ to be a smooth toroidal compactification of $\mathcal{A}_g$ as defined by Faltings and Chai [FC90]. See the survey of Hulek and Tommasi [HT18].

5.5 Open questions: complete subvarieties

Question 5.5.1. *If $g \geq 3$, what is the maximum dimension of a complete subspace of $\mathcal{M}_g$?*

It is possible that the answer to Question 5.5.1 depends on the characteristic.

The answer to this question is at least one because of the following result.

Theorem 5.5.2. *[GDH91] If $g \geq 3$, there exists a complete curve in $\mathcal{M}_g$.*

Proof. Construction: Take $E : y^2 = x^3 - 1$ an elliptic curve and $X : y^2 = x^6 - 1$ which has genus 2. The double cover $\tau : X \to E$ is branched above $(0, i)$ and $(0, -i)$. Let r be even. Choose points $Q_1 = 0_E, Q_2, \ldots, Q_r \in E$ such that $Q_i - Q_j$ is not a 2-torsion point. Let $W = \{(P, P +_E Q_2, \ldots, P +_E Q_r) \mid P \in E\}$. Note that $W \subset E^r - \Delta$ and $W \cong E$. Let $T \subset X^r - \Delta$ be the set of points $\vec{x} = (x_1, \ldots, x_r)$ such that $\tau(x_i) = \tau(x_1) +_E Q_i$. Then T is complete and $\dim(T) \geq 1$.

Now take $r = 2(g-3)$. For each point $\vec{x} \in T$, consider the cover $Z \to X$ branched at the r coordinates of $\vec{x}$. By the Riemann–Hurwitz formula, Z has genus g. The curves are not isomorphic (by Riemann's existence theorem). Thus we have produced a complete curve in $\mathcal{M}_g$. □

The first open case of Question 5.5.1 is $g = 4$, because it is not known if there exists a complete surface in $\mathcal{M}_4$.

Chapter 6

Intersection of the Torelli locus with arithmetic strata

6.1 Overview

In this chapter, we work over an algebraically closed field k of positive characteristic p. We take a more geometric approach to the question of which invariants occur for Jacobians of curves.

Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties of dimension g in characteristic p. There are deep results about the stratifications of $\mathcal{A}_g$ by p-rank, Newton polygon, or Ekedahl Oort type; however, there are very few results about how the open Torelli locus intersects these strata.

This leads to a geometric analogue of Question 4.1.1.

Question 6.1.1. *If p is prime and $g \geq 4$, does the open Torelli locus intersect the strata of $\mathcal{A}_g$ by p-rank, Newton polygon, or Ekedahl-Oort type? If so, what are the geometric properties of the intersection?*

The background Section 6.2 in this chapter is important. Section 6.2.1 contains two facts of major significance: the first is that the Newton polygon can only go up under specialization; the second is the purity result about the dimension of the sublocus where the Newton polygon goes up. In Section 6.2.3, we briefly include results about the dimensions of the arithmetic strata in $\mathcal{A}_g$. In Section 6.2.4, we describe how finding curves with an unusual Newton polyon can be viewed as an unlikely intersection problem.

Section 6.3 contains several results about the geometry of the stratifications of the Torelli locus. The proofs of these results rely on information about the boundary $\partial\mathcal{M}_g$.

Section 6.3.3 contains a proof of [FvdG04, Theorem 2.3] by Faber and Van der Geer, about the dimension of the p-rank strata.

In Section 6.3.4, I describe Theorem 6.3.9 which shows that questions about the geometry of the Newton polygon and Ekedahl-Oort strata can be reduced to the case of p-rank 0. This is an inductive result, similar in spirit to earlier results in the literature, but which allows for more flexibility with the Newton polygon and Ekedahl-Oort type.

6.2 Background

6.2.1 Specialization and purity

Many of the techniques used to study the stratifications on $\mathcal{A}_g$ are not available on the Torelli locus. This includes techniques about deformation (Serre-Tate theory and Dieudonné theory) and Hecke operators. This section includes two major facts known about the behavior of the invariants in families.

The first is that the Newton polygon can only go up under specialization. Specifically, building on Grothendieck's specialization theorem, Katz proved the following:

Theorem 6.2.1. *[Kat79] If A is an $\mathbb{F}_p$-algebra. the set of points in* $\mathrm{Spec}(A)$ *at which the Newton polygon goes up is Zariski-closed, and is locally on* $\mathrm{Spec}(A)$ *the zero-set of a finitely generated ideal.*

Theorem 6.2.1 provides a way to study Newton polygons in families. This was used by Koblitz in [Kob75].

The second is a very important tool: the purity result for Newton polygons proved by de Jong and Oort. Here is the exact statement.

Theorem 6.2.2. *(Purity Theorem [dJO00b, Theorem 4.1]) Let (A, m_A) be a Noetherian local ring of characteristic p. Let S be an F-crystal over* $\mathrm{Spec}(A)$. *Assume that the Newton polygon of S is constant over* $\mathrm{Spec}(A)\backslash\{m_A\}$. *Then either* $\dim(A) < 1$ *or the Newton polygon of S is constant over* $\mathrm{Spec}(A)$.

In practice, the purity theorem is used as follows.

Corollary 6.2.3. *Suppose X is a semi-abelian scheme of dimension g defined over a reduced and irreducible scheme V. Suppose the generic geometric fiber of X has Newton polygon ν. Then the sublocus of points of V whose Newton polygon is not ν is either empty or has codimension* 1 *in V.*

More generally, if ν, ν' are symmetric Newton polygons with $\nu' < \nu$, let $d(\nu', \nu)$ denote the number of symmetric Newton polygons ν'' such that $\nu' \leq \nu'' < \nu$ in the partial ordering of symmetric Newton polygons of dimension g. Then Corollary 6.2.3 implies the following:

Corollary 6.2.4. *Suppose X is a semi-abelian variety of dimension g defined over a reduced and irreducible scheme V. Suppose the generic geometric fiber of X has Newton polygon ν. Then the sublocus of points of V whose Newton polygon is ν' is either empty or has codimension at most $d(\nu', \nu)$ in V.*

In general, it is not possible to conclude that the codimension is exactly $d(\nu', \nu)$ in Corollary 6.2.4 because some of the Newton polygons ν'' between ν and ν' may not occur on V.

6.2.2 Notation for the strata

In this section, let ν denote an arithmetic invariant (such as the p-rank, Newton polygon, Ekedahl–Oort type, or a-number).

Definition 6.2.5. Consider a semi-abelian scheme X of relative dimension g over a Deligne–Mumford stack S. Define $S[\nu]$ to be the locally closed reduced substack of S such that for each field $k' \supset k$ and point $s \in S(k')$, then $s \in S[\nu](k')$ if and only if the arithmetic invariant of X_s is ν.

In the literature, the p-rank f stratum is often denoted with a superscript f. For example, $\mathcal{A}_g^f$ and $\mathcal{M}_g^f$ denote the locally closed reduced substacks of $\mathcal{A}_g$ and $\mathcal{M}_g$, respectively, whose geometric points correspond to objects with p-rank f. Similary, $\bar{\mathcal{M}}_g^f := (\bar{\mathcal{M}}_g)^f$.

Remark 6.2.6. Note that $(\bar{\mathcal{M}}_g)^f$ is the p-rank f stratum of $\bar{\mathcal{M}}_g$, while $\overline{(\mathcal{M}_g^f)}$ is the closure of the p-rank f stratum of $\mathcal{M}_g$. The former may be strictly contained in the latter since the latter may contain points representing curves whose p-rank is strictly less than f.

6.2.3 Dimensions of the strata

This section briefly includes information about the dimensions of the strata in $\mathcal{A}_g$. Let $g \geq 1$. The dimension of $\mathcal{A}_g$ is $g(g+1)/2$. Here is some information about the dimensions of the strata plus a partial list of some valuable references.

(A) The p-rank strata:

For $0 \leq f \leq g$, let $\mathcal{A}_g^f$ denote the p-rank f stratum whose points represent curves of genus g and p-rank f. By [NO80], $\mathcal{A}_g^f$ is non-empty and pure of codimension $g - f$ in $\mathcal{A}_g$.

Oort, *Subvarieties of moduli spaces* [Oor74]

Norman and Oort, *Moduli of abelian varieties* [NO80]

(B) Newton polygon strata:

Let ξ be a symmetric Newton polygon of height $2g$. Consider the stratum $\mathcal{A}_g[\xi]$ of $\mathcal{A}_g$ whose points represent principally polarized abelian varieties with Newton polygon ξ. As in [Oor00, 3.3] or [Oor01a, 1.9], define

$$\mathrm{sdim}(\xi) = \#\Delta(\xi),$$

where

$$\Delta(\xi) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leq g,\ (x, y) \text{ on or above } \xi\}.$$

By [Oor01a, Theorem 4.1], the dimension of $\mathcal{A}_g[\xi]$ is

$$\dim(\mathcal{A}_g[\xi]) = \mathrm{sdim}(\xi).$$

By [CO11], $\mathcal{A}_g[\xi]$ is irreducible if ξ is not the supersingular Newton polygon σ_g. This implies that $\mathcal{A}_g^f$ is irreducible, except when $g = 1, 2$ and $f = 0$.

Koblitz *p-adic variation of the zeta-function over families of varieties defined over finite fields*, [Kob75]

Katz, *Slope filtration of F-crystals*, [Kat79]

de Jong and Oort, *Purity of stratification by Newton polygons* [dJO00b]
Chai and Oort, *Monodromy and irreducibility of leaves* [CO11]

(C) Ekedahl-Oort strata:

Let ξ be a symmetric BT_1 group scheme with Ekedahl-Oort type $\nu = [\nu_1, \ldots, \nu_g]$. By [Oor01b, Theorem 1.2], the stratum of $\mathcal{A}_g$ whose points represent abelian varieties with Ekedahl-Oort type ν is locally closed and quasi-affine with dimension $\sum_{i=1}^{g} \nu_i$.

Kraft, *Kommutative algebraische p-Gruppen* [Kra]
Oort, *A stratification of a moduli space of abelian varieties* [Oor01b]
Moonen and Wedhorn, *Discrete invariants of varieties in positive characteristic* [MW04]
Ekedahl and Van der Geer, *Cycle classes of the E-O stratification on the moduli of abelian varieties* [EvdG09]

6.2.4 Unlikely intersections

Oort observed the following in [Oor05, Expectation 8.5.4]. The moduli space $\mathcal{A}_g$ has dimension $g(g+1)/2$. Its supersingular locus $\mathcal{A}_g[\sigma_g]$ has dimension $\lfloor g^2/4 \rfloor$. The difference $\delta_g := g(g+1)/2 - \lfloor g^2/4 \rfloor$ is the length of a chain which connects the ordinary Newton polygon ν_g to the supersingular Newton polygon σ_g in the partially ordered set of Newton polygons of dimension g.

Remark 6.2.7. If $g \geq 9$, then $\delta_g > 3g - 3 = \dim(\mathcal{M}_g)$.

Because of Remark 6.2.7, at least one of the following is true:

1. Either $\mathcal{M}_g$ does not admit a perfect stratification by Newton polygon: this means that there are two Newton polygons ξ_1 and ξ_2 such that $\mathcal{A}_g[\xi_1]$ is in the closure of $\mathcal{A}_g[\xi_2]$, but $\mathcal{M}_g[\xi_1]$ is not in the closure of $\mathcal{M}_g[\xi_2]$;

2. or some Newton polygons do not occur for Jacobians of smooth curves.

At this time, no Newton polygon has been excluded from occurring for a Jacobian in any characteristic.

Definition 6.2.8. Let η denote a Newton polygon or Ekedahl–Oort type in dimension g. We say that $\mathcal{M}_g$ and $\mathcal{A}_g[\eta]$ have an unlikely intersection if $\mathrm{codim}(\mathcal{A}_g[\eta], \mathcal{A}_g) > 3g - 3$.

From Section 4.4.3, which includes constructions of supersingular curves for arbitrarily high genus, it is clear that unlikely intersections do occur.

In fact, [Oor05, Conjecture 8.5.7] predicts that Newton polygons having small denominators will always occur for Jacobians of smooth curves.

6.3 Main theorems

In this section, we describe several results about the geometry of the stratifications of the Torelli locus.

Let $\mathcal{M}_g$ denote the moduli space of smooth curves of genus g in characteristic p. Via the Torelli morphism, the moduli space $\mathcal{M}_g$ also has stratifications by the arithmetic invariants.

A careful analysis of the boundary of $\mathcal{M}_g$ gives results about Question 6.1.1 for the p-rank strata. The proofs of these results rely on information about the boundary $\partial\mathcal{M}_g$. It is important to keep in mind that the Torelli morphism is not flat since the fibers have positive dimension over $\partial\mathcal{M}_g$.

6.3.1 Invariants of stable curves

By Definition 6.2.5, we denote by $\Delta_i[\bar{\mathcal{M}}_g][\nu]$ the sublocus of $\Delta_i[\bar{\mathcal{M}}_g]$ representing curves with invariant ν.

Recall that the generic geometric point of Δ_i represents a curve D with two irreducible components C_1 and C_2, having genera $g_1 = i$ and $g_2 = g - i$, that intersect in an ordinary double point. By (5.2), $\mathrm{Jac}(D) \simeq \mathrm{Jac}(C_1) \oplus \mathrm{Jac}(C_1)$, so the p-rank, Newton polygon, and p-torsion group scheme of D are the sum of those of C_1 and C_2.

Recall that the generic geometric point of Δ_0 represents a curve with one irreducible component that self-intersects in an ordinary double point. The p-rank of a semi-abelian variety A is $f = \dim_{\mathbb{F}_p}\mathrm{Hom}(\mu_p, A)$. It follows from (5.3) that the torus $W \to \mathrm{Pic}^0(\tilde{C})$ increases the p-rank by 1. This increases the multiplicity of the slopes 0 and 1 in the Newton polygon by one and increases the multiplicity of $\mathbb{Z}/p\mathbb{Z} \oplus \mu_p$ by one in the p-torsion group scheme. The Ekedahl–Oort type of a stable curve is defined in two different ways in [EvdG09] and [Moo22]; these are proven to agree in [Draa].

6.3.2 A geometric proof for supersingular genus 4 curves

This result was inspired by a conversation with Oort, in which we discussed a more geometric method for studying the Newton polygons that occur on $\mathcal{M}_g$. This method applies when the codimension of the Newton polygon stratum in $\mathcal{A}_g$ is small.

As an illustration of this method, here is a new proof of [KHS20, Corollary 1.2]. Let $\mathcal{M}_g[ss]$ (resp. $\mathcal{A}_g[ss]$) denote the supersingular locus of $\mathcal{M}_g$ (resp. $\mathcal{A}_g$).

Theorem 6.3.1. *[Pri] For every prime p, there exists a smooth curve of genus* 4 *that is supersingular. Thus* $\mathcal{M}_4[ss]$ *is non-empty and its irreducible components have dimension at least* 3 *for every prime p.*

This method does not give a new proof of [KHS20, Theorem 1.1], which states that there exists a supersingular smooth curve of genus 4 with a-number $a \geq 3$ for every prime $p > 3$.

Proof of Theorem 6.3.1. Over $\overline{\mathbb{F}}_p$, there exists a stable curve C of genus 4 that is singular and supersingular. For example, this can be produced by taking a chain of four supersingular elliptic curves, clutched together at ordinary double points. This yields a curve of compact type. So the Jacobian of C is a principally polarized abelian variety of dimension 4. Furthermore, the Jacobian is isomorphic to the product of four supersingular elliptic curves and thus is supersingular. As such, it is represented by a point in $\mathcal{A}_4[ss] \cap T_4$, where T_4 is the locus of Jacobians of stable curves of genus 4.

The codimension of $\mathcal{A}_4[ss]$ in $\mathcal{A}_4$ is $10 - 4 = 6$. The codimension of $T_4 \cap \mathcal{A}_4$ in $\mathcal{A}_4$ is $10 - 9 = 1$. Since $\mathcal{A}_4$ is a smooth stack, the codimension of an intersection of two substacks is at most the sum of their codimensions [Vis89, page 614]. Thus $\mathrm{codim}(\mathcal{A}_4[ss] \cap T_4, \mathcal{A}_4) \leq 7$.

To summarize, $\mathcal{A}_4[ss] \cap T_4$ is non-empty and each of its irreducible components has dimension at least 3.

Let δ denote the locus in $\mathcal{A}_4[ss] \cap T_4$ whose points represent the Jacobian of a curve C_s that is stable but not smooth. Since the Jacobian is an abelian variety, the curve C_s has compact type. So its Jacobian is a principally polarized abelian fourfold that decomposes, with the product polarization.

Then $\dim(\delta) \leq 2$. This is because points in δ parametrize objects either of the form $E \oplus X$ where E is a supersingular elliptic curve and X is a supersingular abelian threefold, or of the form $X \oplus X'$ where X, X' are supersingular abelian surfaces. In the former case, the dimension is $\dim(\mathcal{A}_1[ss] \oplus \mathcal{A}_3[ss]) = 0 + 2 = 2$. In the latter case, the dimension is $\dim(\mathcal{A}_2[ss] \oplus \mathcal{A}_2[ss]) = 1 + 1 = 2$. Since $2 < 3$, every generic geometric point of $\mathcal{A}_4[ss] \cap T_4$ represents the Jacobian of a supersingular curve of genus 4 which is smooth.

Thus $\mathcal{M}_4[ss]$ is non-empty for every p; this is equivalent to the statement that there exists a smooth curve of genus 4 that is supersingular. If R is an irreducible component of $\mathcal{M}_4[ss]$, then the image of R under the Torelli morphism is open and dense in a component of $\mathcal{A}_4[ss] \cap T_4$; so $\dim(R) \geq 3$, which completes the proof. □

Remark 6.3.2. One expects that the dimension of every component of $\mathcal{M}_4[ss]$ is three. For $7 < p < 20,000$ or $p \equiv 5 \bmod 6$, this is true for at least one component of $\mathcal{M}_4[ss]$ by [Har22, Theorem 2.4, Corollary 4.4]. It is true for every component when $p = 2$ in [Drab], and when $p = 3$, as a consequence of [Draa, Theorem C].

6.3.3 Results about the p-rank stratification

In this section, we describe a theorem of Faber and Van der Geer that the p-rank strata have the expected dimension in the moduli space $\mathcal{M}_g$ of curves of genus g. Fix a prime p and integers $g \geq 2$ and f such that $0 \leq f \leq g$.

The moduli space $\mathcal{M}_g$ can be stratified by p-rank into strata $\mathcal{M}_g^f$ whose points represent curves of genus g and p-rank f. Similarly, one can stratify the moduli space $\mathcal{H}_g$ of hyperelliptic curves or the compactifications $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{H}}_g$ by p-rank.

Recall that $\mathcal{A}_g^f$ is irreducible unless $g = 1, 2$ and $f = 0$. In most cases, it is not known whether $\mathcal{M}_g^f$ and $\mathcal{H}_g^f$ are irreducible.

Theorem 6.3.3. *[FvdG04, Theorem 2.3] Let $g \geq 2$. Every component of $\overline{\mathcal{M}}_g^f$ has dimension $2g - 3 + f$ (codimension $g - f$ in $\overline{\mathcal{M}}_g$); in particular, there exists a smooth curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.*

Theorem 6.3.4. *(p odd) [GP05, Theorem 1], see also [AP11, Lemma 3.1], ($p = 2$) [PZ12, Corollary 1.3] Every component of $\overline{\mathcal{H}}_g^f$ has dimension $g - 1 + f$ (codimension $g - f$ in $\overline{\mathcal{H}}_g$); in particular, there exists a smooth hyperelliptic curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.*

Remark 6.3.5. In [AP08] and [AP11], the authors prove more about the components of $\overline{\mathcal{M}}_g^f$ and $\overline{\mathcal{H}}_g^f$; this includes results about how the components intersect the boundary and results about the ℓ-adic monodromy of the components. In [Pri09], for all $g \geq 3$ and all p, there are results about the moduli of curves with p-rank $g - 2$ or $g - 3$ and a-number $a \geq 2$.

We give a sketch of the proof of Theorem 6.3.3; it uses the boundary of $\overline{\mathcal{M}}_g$.

By Section 6.3.1, the p-rank of a singular curve of compact type is the sum of the p-ranks of its components. Thus, it is easy to construct a *singular* curve of genus g with p-rank f, by constructing a chain of f ordinary and $g-f$ supersingular elliptic curves, joined at ordinary double points. This singular curve can be deformed to a smooth one, but it is not obvious that the p-rank stays constant in this deformation. To prove that there is a *smooth* curve of genus g with p-rank f, singular curves are still useful, but the argument must be made more carefully.

Recall that $\overline{\mathcal{M}}_{g;1}$ is the moduli space whose points represent curves C of genus g together with a marked point x. The dimension of $\overline{\mathcal{M}}_{g;1}$ is $3g-3+1$ for all $g \geq 1$. Recall the clutching morphism $\kappa_{i,g-i}$ from Section 5.2.1.

Proof. (Sketch of proof of Theorem 6.3.3) The proof is by induction on g. When $g = 2, 3$, the result is true since the open Torelli locus is open and dense in $\mathcal{A}_g$. Suppose $g \geq 4$.

The dimension of $\overline{\mathcal{M}}_g$ is $3g-3$. There are singular curves that are ordinary, namely chains of g ordinary elliptic curves. Since $\overline{\mathcal{M}}_g$ is irreducible and the p-rank is lower semi-continuous, the generic geometric point of $\overline{\mathcal{M}}_g$ is ordinary, with p-rank g.

Let S be a component of $\overline{\mathcal{M}}_g^f$. The length of the chain which connects the ordinary Newton polygon ν_g to the largest Newton polygon having $(f, 0)$ as a break point is $g - f$. Using purity of the Newton polygon stratification [dJO00b],

$$\dim(S) \geq (3g-3) - (g-f) = 2g - 3 + f.$$

By [FvdG04, Lemma 2.5], S intersects Δ_i for each $1 \leq i \leq g-1$. By Theorem 5.2.3, $\mathrm{codim}(\Delta_i, \overline{\mathcal{M}}_g) = 1$. It follows from [Vis89, page 614] that $\dim(S) \leq \dim(S \cap \Delta_i) + 1$.

The p-rank of a singular curve of compact type is the sum of the p-ranks of its components, [BLR90, Example 8, Page 246]. As seen in [AP08, Proposition 3.4], one can restrict the clutching morphism to the p-rank strata:

$$\kappa_{i,g-i} : \overline{\mathcal{M}}_{i;1}^{f_1} \times \overline{\mathcal{M}}_{g-i;1}^{f_2} \to \overline{\mathcal{M}}_g^{f_1+f_2}.$$

This means that $\dim(S \cap \Delta_i)$ is bounded above by $\dim(\overline{\mathcal{M}}_{i;1}^{f_1}) + \dim(\overline{\mathcal{M}}_{g-i;1}^{f_2})$, for some pair (f_1, f_2) such that $f_1 + f_2 = f$. Adding a marked point adds one to the dimension. By the inductive hypothesis (or an explicit computation when $i = 1, g-1$), one checks that $\dim(\overline{\mathcal{M}}_{i;1}^{f_1}) = 2i - 3 + f_1 + 1$ and $\dim(\overline{\mathcal{M}}_{g-i;1}^{f_2}) = 2(g-i) - 3 + f_2 + 1$. It follows that $\dim(S \cap \Delta_i) \leq 2g - 4 + f$. Thus $\dim(S) \leq 2g - 3 + f$, which completes the proof. □

6.3.4 Increasing the p-rank

This section contains an inductive result. Starting with a Newton polygon ξ that can be realized for a smooth curve of genus g, the goal is to prove that any symmetric Newton polygon which is formed by adjoining slopes of 0 and 1 to ξ can also be realized for a smooth curve (of larger genus and p-rank). I show this is possible under a geometric condition on the stratum of $\mathcal{M}_g$ with Newton polygon ξ.

The importance of this result is that it allows us to restrict to the case of p-rank 0 in Question 6.1.1. This type of inductive process can be found in earlier work, e.g., [FvdG04, Theorem 2.3], [AP08, Section 3], [Pri09, Proposition 3.7], and [AP14, Proposition 5.4]. Theorem 6.3.9 is stronger than these results because it allows for more flexibility with the Newton polygon and Ekedahl-Oort type.

First, we fix some notation about Newton polygons and BT_1 group schemes.

Notation 6.3.6. Let ξ denote a symmetric Newton polygon (or a symmetric BT_1 group scheme) occurring for principally polarized abelian varieties in dimension g. Let $\mathcal{A}_g[\xi]$ be the stratum in $\mathcal{A}_g$ whose geometric points represent principally polarized abelian varieties of dimension g and type ξ. Let $cd_\xi = \mathrm{codim}(\mathcal{A}_g[\xi], \mathcal{A}_g)$. Let $\mathcal{M}_g[\xi]$ be the stratum in $\mathcal{M}_g$ whose geometric points represent smooth projective curves of genus g and type ξ.

Notation 6.3.7. In the case that ξ denotes a symmetric Newton polygon occurring in dimension g: for $e \in \mathbb{N}$, let ξ^{+e} be the symmetric Newton polygon in dimension $g+e$ such that the difference between the multiplicity of the slope λ in ξ^{+e} and the multiplicity of the slope λ in ξ is 0 if $\lambda \notin \{0,1\}$ and is e if $\lambda \in \{0,1\}$.

Notation 6.3.8. In the case that ξ denotes a symmetric BT_1 group scheme occurring in dimension g: for $e \in \mathbb{N}$, let ξ^{+e} be the symmetric BT_1 group scheme in dimension $g+e$ given by

$$\xi^{+e} := L^e \oplus \xi,$$

where $L = \mathbb{Z}/p \oplus \mu_p$. If $[\nu_1, \ldots, \nu_g]$ is the Ekedahl-Oort type of ξ, then ξ^{+e} has Ekedahl-Oort type $[1, 2, \ldots, e, \nu_1 + e, \ldots, \nu_g + e]$.

Theorem 6.3.9. *[Pri19, Theorem 6.4] With notation as in 6.3.6, 6.3.7, 6.3.8, suppose that there exists an irreducible component $S = S_0$ of $\mathcal{M}_g[\xi]$ such that* $\mathrm{codim}(S, \mathcal{M}_g) = cd_\xi$. *Then, for all $e \in \mathbb{N}$, there exists a component S_e of $\mathcal{M}_{g+e}[\xi^{+e}]$ such that* $\mathrm{codim}(S_e, \mathcal{M}_{g+e}) = cd_\xi$.

The proof uses the boundary component Δ_1. A similar result using the boundary component Δ_0 can be found in [Draa].

6.4 Related results

Here are some applications of these methods:

Corollary 6.4.1. *[Pri, Corollary 4.3] For every prime p, every symmetric Newton polygon in dimension g having p-rank $f \geq g - 4$ occurs on $\mathcal{M}_g$.*

Corollary 6.4.2 (Dragutinović and Pries)**.** *For every prime p, there exists a smooth curve of genus g with p-rank* 0 *and a-number at least* 2*.*

Corollary 6.4.3. *[Draa, Corollary 6.4] When $p = 2$, for every $g \geq 4$, there exists a smooth curve with p-rank $f = g - 3$ and Young type $\{3, 2\}$.*

6.5 Open questions

Suppose η is a Newton polygon or Ekedahl–Oort type which occurs on $\mathcal{M}_g$ in characteristic p, meaning that there exists a smooth curve of genus g defined over $\bar{\mathbb{F}}_p$ having type η. Even so, there are open questions. In this section, we describe open questions about the number of components of the strata and about the statistical behavior of the number of these curves.

The questions in this section can be asked for almost all Newton polygons and Ekedahl–Oort types, for almost all values of g. To make the questions more tractable, we focus on particular cases in which the answer is not known. More information about these questions will be provided later.

6.5.1 Number of components of the strata

If η is a Newton polygon that is not supersingular, then the locus $\mathcal{A}_g[\eta]$ is irreducible. Similarly, if η is an Ekedahl–Oort type that is not fully contained in $\mathcal{A}_g[\mathrm{ss}^g]$, then the locus $\mathcal{A}_g[\eta]$ is irreducible.

However, in most cases, the number of components in the intersection $\mathcal{A}_g[\eta] \cap \mathcal{T}_g^\circ$ is not known.

For example, let η denote the almost ordinary Newton polygon, namely $\eta = \mathrm{o}^{g-1} \oplus \mathrm{ss}$. In other words, the Newton polygon η has $g-1$ slopes of 0, two slopes of 1/2, and $g-1$ slopes of 1. There is a unique Ekedahl–Oort type for η, which is $(\mathbb{Z}/p\mathbb{Z} \oplus \mu_p)^{g-1} \oplus I_{1,1}$.

The non-ordinary locus of $\mathcal{A}_g \cap \mathcal{T}_g^\circ$ is closed of codimension 1 in $\mathcal{A}_g \cap \mathcal{T}_g^\circ$. It has dimension $3g-4$, but it is not known whether it is irreducible in general.

Question 6.5.1. *Let $g \geq 4$. Let $\eta = \mathrm{o}^{g-1} \oplus \mathrm{ss}$ denote the almost ordinary Newton polygon. What is the number of components in the intersection $\mathcal{A}_g[\eta] \cap \mathcal{T}_g^\circ$?*

Question 6.5.1 is equivalent to asking for the number of components of the non-ordinary locus of $\mathcal{M}_g$ or of the p-rank $g-1$ strata in $\mathcal{M}_g$.

Example 6.5.2. When $g = 2$ (resp. $g = 3$), the answer to Question 6.5.1 is 1.

A curve C is non-ordinary if and only if the matrix for V on $H^0(C, \Omega^1)$ has determinant 0. Because the entries of this matrix increase in complexity with p, it is difficult to solve Question 6.5.1 algebraically.

6.5.2 A statistical approach

Question 6.5.3. *Given p prime and $g \geq 4$ an integer: Let $q = p^a$ be a power of p. Let η denote the almost ordinary Newton polygon. What is the order of magnitude of $\mathcal{M}_g[\eta](\mathbb{F}_q)$, in terms of p, g, and a?*

This question is already interesting for $g = 4$.

Remark 6.5.4. For p and a sufficiently large, one expects that the answer to this question is of the form $Cp^{a(3g-4)}$, for some constant C. Here one guesses that C depends on g but not on a. It is not clear whether C is independent of p. Using an arithmetic statistics approach, the value of C gives information about Question 6.5.1.

Example 6.5.5. Look at $y^m = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}$. Let a_4 be such that $\sum_{i=1}^4 a_i \equiv 0 \bmod m$. This is a one-dimensional family of curves that are a cyclic degree m cover of $\mathbb{P}^1$. Suppose the curve is ordinary for a typical choice of t. This happens if $p \equiv 1 \bmod m$ or if $a_1 + a_2 = m$. In this situation, Cavalieri and I found a mass formula for the number of non-ordinary curves in the family [CP, Corollary 6.1] The formula depends on the a-numbers of curves that are not ordinary in the family. More information can be given when the family is special; see Example 8.4.2.

6.5.3 Intersection of the supersingular locus with the boundary

Question 6.5.6. *Determine the intersection of the supersingular locus of $\mathcal{M}_3$ with the boundary of $\mathcal{M}_3$; similar question for the hyperelliptic locus $\mathcal{H}_3$. Generalize to $\mathcal{M}_4$.*

6.5.4 Double covers of an elliptic curve

Question 6.5.7. *Study the dimensions of the p-rank strata of the moduli space of double covers of a fixed elliptic curve with $2n$ branch points.*

Chapter 7

Curves and abelian varieties with cyclic action

7.1 Overview

In this chapter, we focus on curves C and abelian varieties X that have an automorphism of order m.

Specifically, we consider curves C that are cyclic branched covers of the projective line. The moduli spaces for these covers of curves are called Hurwitz spaces. The irreducible components of the Hurwitz spaces are indexed by monodromy data, which includes the data for the cover, including the degree m, the number of branch points N, and the inertia type a. The dimension of each component of the Hurwitz space is $N-3$.

We consider abelian varieties X having an automorphism of order m, with the restriction that the trivial eigenspace for the μ_m-action is zero. The moduli spaces for these abelian varieties are called Deligne–Mostow Shimura varieties.

Using a generalization of the Torelli morphism, it is possible to map the Hurwitz spaces to the Shimura varieties. When the image is open and dense in a component of the Shimura variety, the family is called *special*.

7.2 Background

Let C be a cyclic branched cover of the projective line. Let m be the degree of the cover. We assume throughout this chapter that $\mathrm{char}(k) \nmid m$. Let $\tau \in \mathrm{Aut}(C)$ be an automorphism of order m such that $C/\langle\tau\rangle \simeq \mathbb{P}^1$.

7.2.1 Equations of cyclic covers of the projective line

Lemma 7.2.1. *Suppose C is a curve that admits a μ_m-cover $\phi : C \to \mathbb{P}^1$. Let N be the number of branch points of ϕ. Then C has an equation of the form*

$$y^m = \prod_{i=1}^{N}(x-b_i)^{a_i}, \tag{7.1}$$

for some distinct values $b_1, \ldots, b_N \in k$ *and some integers* $a_1, \ldots, a_N$ *such that* $1 \le a_i < m$ *and* $\sum_{i=1}^N a_i \equiv 0 \bmod m$. *Also, a given automorphism* τ *of order* m *acts by* $\tau((x,y)) = (x, \zeta_m y)$.

Proof. By Kummer theory, there is an affine equation for C of the form $y^m = f(x)$, for some rational function $f(x) \in k(x)$. After some changes of coordinates, we can suppose that $f(x) \in k[x]$ is a polynomial and that each root of $f(x)$ has order less than m. Then the roots of $f(x)$ are the branch points and we label these as $b_1, \ldots, b_N$. After a fractional linear transformation, where, without loss of generality, we suppose that $b_1 = 0$, $b_2 = 1$ and $b_N = \infty$. Then there are integers $a_1, \ldots, a_N$ such that $1 \le a_i < m$ such that (7.1) is satisfied. The fact that $\sum_{i=1}^N a_i \equiv 0 \bmod m$ comes from the topological description of the fundamental group of $X - B$. □

Definition 7.2.2. Fix integers $m \ge 2, N \ge 3$ and an N-tuple of positive integers $a = (a_1, \ldots, a_N)$. Then a is an *inertia type* for m and (m, N, a) is a *monodromy datum* if

1. $a_i \not\equiv 0 \bmod m$, for each $1 \le i \le N$,
2. $\gcd(m, a_1, \ldots, a_N) = 1$, and
3. $\sum_{i=1}^N a_i \equiv 0 \bmod m$.

Fix a monodromy datum (m, N, a). Let $U \subset (\mathbb{A}^1)^N$ be the locus of points where no two of the coordinates are equal. Over U, we can define a curve C to be the smooth projective (relative) curve whose fiber at each point $b = (b_1, \ldots, b_N) \in U$ has affine model

$$y^m = \prod_{i=1}^N (x - b_i)^{a_i}. \tag{7.2}$$

The function x on C yields a map $C \to \mathbb{P}^1_U$ and there is a μ_m-action on C over U given by $\zeta \cdot (x, y) = (x, \zeta \cdot y)$ for all $\zeta \in \mu_m$. Thus $C \to \mathbb{P}^1_U$ is a μ_m-cover.

Alternatively, if the field of definition of C is sufficiently large, one can move three of the branch points to $0, 1, \infty$. Then we take $U \subset (\mathbb{A}^1 - \{0, 1\})^{N-3}$ to be the locus of points where no two of the coordinates are equal. In that case, (7.2) simplifies to:

$$y^m = x^{a_1}(x-1)^{a_2} \prod_{i=3}^{N-1} (x - b_i)^{a_i}. \tag{7.3}$$

For a closed point $t \in U$, let C_t denote the smooth projective curve with affine equation (7.2) (or (7.3)). There is a μ_m-cover $C_t \to \mathbb{P}$ taking $(x, y) \mapsto x$; it is branched at N points $b_1, \ldots, b_N$ in $\mathbb{P}^1$, and has local monodromy a_i at b_i. Let J_t be the Jacobian of C_t.

Remark 7.2.3. If $a_i > 1$, then the affine curve has a singularity at the point $(b_i, 0)$. Finding the equation for the blow-up is a long process and is best avoided.

7.2.2 The genus and the signature

Lemma 7.2.4. *[Riemann–Hurwitz formula] For all $t \in U$, the curve C_t is irreducible. Its genus g is $(m-1)(N-2)/2$ if m is prime. More generally, the genus is:*

$$g = g(m, N, a) = 1 + \frac{(N-2)m - \sum_{i=1}^{N} \gcd(a(i), m)}{2}. \tag{7.4}$$

The Jacobian J_t and all the cohomology groups of C_t are modules for the group ring $\mathbb{Z}[\mu_m]$. We would like to determine how they decompose into eigenspaces under the μ_m-action. This calculation can be done over $\mathbb{C}$. Let V be the first Betti cohomology group $H^1(C_t(\mathbb{C}), \mathbb{Q})$. Let $V^+ = H^0(C_t(\mathbb{C}), \Omega^1_{C_t})$.

Recall that we fixed an mth root of unity $\zeta_m \in \mu_m$. The data of a μ_m-cover includes an inclusion of μ_m in $\mathrm{Aut}(C_t)$. There is an induced action of μ_m on V^+. For $0 \leq n \leq m-1$, let L_n denote the subspace of $\omega \in V^+$ such that $\zeta_m \cdot \omega = \zeta_m^n \omega$. The subspace L_0 is trivial since C_t is a μ_m-cover of $\mathbb{P}^1$. There is a decomposition:

$$V^+ = \oplus_{1 \leq n \leq m-1} L_n.$$

Let $\mathfrak{f}_n = \dim(L_n)$. Note that $\sum_{n=1}^{m-1} = g$. The dimension $\mathfrak{f}_n$ is independent of the choice of $t \in U$.

For any $q \in \mathbb{Q}$, let $\langle q \rangle$ denote the fractional part of x.

Lemma 7.2.5 (Hurwitz, Chevalley-Weil)**.** *see [Moo10, Lemma 2.7, §3.2] If $1 \leq n \leq m-1$, then*

$$\mathfrak{f}_n = -1 + \sum_{i=1}^{N} \langle \frac{-na(i)}{m} \rangle \tag{7.5}$$

Definition 7.2.6. The *signature type* of the monodromy datum (m, N, a) is

$$\mathfrak{f} = (\mathfrak{f}_1, \ldots, \mathfrak{f}_{m-1}).$$

7.2.3 Hurwitz spaces

Let $\gamma = (m, N, a)$ be a monodromy datum with $N \geq 4$. The Hurwitz space H_γ is the moduli space of μ_m-covers $\phi : C \to \mathbb{P}^1$ having monodromy datum γ. There is a forgetful map $H_\gamma \to \mathcal{M}_g$ that takes the isomorphism class of ϕ to the isomorphism class of C.

Theorem 7.2.7. *[Ful69, Corollary 7.5], [Wew98, Corollary 4.2.3] The Hurwitz space H_γ is irreducible. It has dimension* $\dim(H_\gamma) = N - 3$.

7.3 Main theorems

We would like to understand the subspace of $\mathcal{A}_g$ whose points represent Jacobians of curves that are cyclic covers of $\mathbb{P}^1$. In this section, we take a more accessible approach to this topic. In the next section, we approach the same topic from the perspective of unitary Shimura varieties.

Let $\gamma = (m, N, a)$ be a monodromy datum with $N \geq 4$, let g be the associated genus given by Lemma 7.2.4, and let $\mathfrak{f}$ be the associated signature type given by (7.5).

Given an μ_m-cover $C \to \mathbb{P}^1$ with monodromy datum γ, then the Jacobian $\mathrm{Jac}(C)$ is a p.p. abelian variety of dimension g, with an induced action of the group ring $\mathbb{Z}[\mu_m]$, such that the signature of the action is given by $\mathfrak{f}$.

The composition of the Torelli map yields a morphism

$$j = j_\gamma : H_\gamma \to \mathcal{M}_g \to \mathcal{A}_g.$$

Definition 7.3.1. If $\gamma = (m, N, a)$ is a monodromy datum, let T°_γ be the image of j_γ in $\mathcal{A}_g$ (with the reduced induced structure). Let T_γ be the closure of T°_γ in $\mathcal{A}_g$.

By definition, T_γ is a closed, reduced substack of $\mathcal{A}_g$.

Remark 7.3.2. Suppose $\phi : C \to \mathbb{P}^k$ is a μ_m-cover with monodromy datum γ. Changing the generator of μ_m does not change C or $\mathrm{Jac}(C)$. Changing the order of the branch points does not change C or $\mathrm{Jac}(C)$. So T_γ depends uniquely on the equivalence class of the monodromy datum $\gamma = (m, N, a)$, where (m, N, a) and (m', N', a') are equivalent if $m = m'$, $N = N'$, and the images of a, a' in $(\mathbb{Z}/m\mathbb{Z})^N$ are in the same orbit under $(\mathbb{Z}/m\mathbb{Z})^* \times \mathrm{Sym}_N$.

Another way to think about T_γ is this. Consider the subspace of $\mathcal{A}_g$ whose points represent p.p. abelian varieties having an action by the group ring $\mathbb{Z}[\mu_m]$, with signature $\mathfrak{f}$. Then T_γ is the intersection of that subspace with the Torelli locus.

That subspace is essentially the image of a Shimura variety. Naively speaking, we are going to look at the moduli space of abelian varieties of dimension g, equipped with an action of $\mathbb{Z}[\mu_m]$, with the signature of the action given by $\mathfrak{f}$.

In [DM86] Deligne and Mostow construct the smallest unitary Shimura variety whose image in $\mathcal{A}_g$ contains T_γ; we denote it by $S_\gamma = \mathrm{Sh}(\mu_m, \mathfrak{f})$. Section 7.4 contains the basic definitions and facts about PEL-type Shimura varieties, and the construction of [DM86], following [Moo10].

Here is a schematic diagram of the moduli spaces:

$$\begin{array}{ccc} H_\gamma & \xrightarrow{\varphi} & S_\gamma \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \xrightarrow{\tau_g} & T_\gamma \qquad \subset \mathcal{A}_g. \end{array} \tag{7.6}$$

The main result we will need is the dimension of S, which is given as follows.

Proposition 7.3.3. *[MO13, Proposition 5.13] Let $\gamma = (m, N, a)$ be a monodromy datum with associated signature $\mathfrak{f}$. If $m = 2k$ is even, let $\epsilon_\gamma = \mathfrak{f}_k(\mathfrak{f}_k + 1)/2$; if m is odd, let $\epsilon_\gamma = 0$. Then the dimension of the Shimura variety $S_\gamma = \mathrm{Sh}(\mu_m, \mathfrak{f})$ is*

$$\dim(S_\gamma) = \epsilon_\gamma + \sum_{n=1}^{\lfloor m/2 \rfloor} \mathfrak{f}_n \mathfrak{f}_{-n}. \tag{7.7}$$

The proof of this result goes beyond the scope of these notes. The main ideas are to look at the Hodge structure and symplectic form and to compute the dimension of the tangent space.

7.4 Related results - Shimura varieties

This section is more technical and can be skipped.

7.4.1 Shimura datum for the moduli space of abelian varieties

Let $V = \mathbb{Q}^{2g}$, and let $\Psi : V \times V \to \mathbb{Q}$ denote the standard symplectic form. Let $G := \mathrm{GSp}(V, \Psi)$ denote the group of symplectic similitudes over $\mathbb{Q}$. Let $\mathfrak{h}$ denote the space of homomorphisms $h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ which define a Hodge structure of type $(-1, 0) + (0, -1)$ on $V_{\mathbb{Z}}$ such that $\pm(2\pi i)\Psi$ is a polarization on V. The pair $(G, \mathfrak{h})$ is the Shimura datum for $\mathcal{A}_g$.

Let $H \subset G$ be an algebraic subgroup over $\mathbb{Q}$ such that the subspace

$$\mathfrak{h}_H := \{h \in \mathfrak{h} \mid h \text{ factors through } H_{\mathbb{R}}\}$$

is non-empty. Then $H(\mathbb{R})$ acts on $\mathfrak{h}_H$ by conjugation, and for each $H(\mathbb{R})$-orbit $Y_H \subset \mathfrak{h}_H$, the Shimura datum (H, Y_H) defines an algebraic substack $\mathrm{Sh}(H, Y_H)$ of $\mathcal{A}_g$. In the following, for $h \in Y_H$, we sometimes write (H, h) for the Shimura datum (H, Y_H). For convenience, we also write $\mathrm{Sh}(H, \mathfrak{h}_H)$ for the finite union of the Shimura stacks $\mathrm{Sh}(H, Y_H)$, as Y_H varies among the $H(\mathbb{R})$-orbits in $\mathfrak{h}_H$.

7.4.2 Shimura data of PEL-type

Now we focus on Shimura data of PEL-type. Let B be a semisimple $\mathbb{Q}$-algebra, together with an involution $*$. Suppose there is an action of B on V such that $\Psi(bv, w) = \Psi(v, b^* w)$, for all $b \in B$ and all $v, w \in V$. Let

$$H_B := \mathrm{GL}_B(V) \cap \mathrm{GSp}(V, \Psi).$$

We assume that $\mathfrak{h}_{H_B} \neq \emptyset$.

For each $H_B(\mathbb{R})$-orbit $Y_B := Y_{H_B} \subset \mathfrak{h}_{H_B}$, the associated Shimura stack $\mathrm{Sh}(H_B, Y_B)$ arise as moduli spaces of polarized abelian varieties endowed with a B-action, and are called of PEL-type. In the following, we also write $\mathrm{Sh}(B) := \mathrm{Sh}(H_B, \mathfrak{h}_{H_B})$.

Each homomorphism $h \in Y_B$ defines a decomposition of $B_{\mathbb{C}}$-modules

$$V_{\mathbb{C}} = V^+ \oplus V^-$$

where V^+ (respectively, V^-) is the subspace of $V_{\mathbb{C}}$ on which $h(z)$ acts by z (respectively, by $\bar{z}$). The isomorphism class of the $B_{\mathbb{C}}$-module V^+ depends only on Y_B. Moreover, Y_B is determined by the isomorphism class of V^+ as a $B_{\mathbb{C}}$-submodule of $V_{\mathbb{C}}$. In the following, we prescribe Y_B in terms of the $B_{\mathbb{C}}$-module V^+. By construction, $\dim_{\mathbb{C}} V^+ = g$.

7.4.3 Shimura subvariety attached to a monodromy datum

We consider cyclic covers of the projective line branched at more than three points; fix a monodromy datum (m, N, a) with $N \geq 4$. Take $B = \mathbb{Q}[\mu_m]$ with involution $*$.

As in Section 7.2.1, let $C \to U$ denote the universal family of μ_m-covers of $\mathbb{P}^1$ branched at N points with inertia type a; let $j = j(m, N, a) : U \to \mathcal{A}_g$ be the composition of the Torelli map with the morphism $U \to \mathcal{M}_g$. From Definition 7.3.1, recall that $Z = Z(m, N, a)$ is the closure in $\mathcal{A}_g$ of the image of $j(m, N, a)$.

The pullback of the universal abelian scheme $\mathcal{X}$ on $\mathcal{A}_g$ via j is the relative Jacobian $\mathcal{J}$ of $C \to U$. Since μ_m acts on C, there is a natural action of the group algebra $\mathbb{Z}[\mu_m]$ on $\mathcal{J}$. We also use $\mathcal{J}$ to denote the pullback of $\mathcal{X}$ to Z. The action of $\mathbb{Z}[\mu_m]$ extends naturally to $\mathcal{J}$ over Z. Hence the substack $Z = Z(m, N, a)$ is contained in $\mathrm{Sh}(\mathbb{Q}[\mu_m])$ for an appropriate choice of a structure of $\mathbb{Q}[\mu_m]$-module on V. More precisely, fix $x \in Z(\mathbb{C})$, and let $(\mathcal{J}_x, \theta)$ denote the corresponding Jacobian with its principal polarization θ. Choose a symplectic similitude, meaning an isomorphism

$$\alpha : (H_1(\mathcal{J}_x, \mathbb{Q}), \psi_\theta) \to (V, \Psi),$$

such that the pull back of the symplectic form Ψ to $H_1(\mathcal{J}_x, \mathbb{Q})$ is a scalar multiple of ψ_θ, where ψ_θ denotes the Riemannian form on $H_1(\mathcal{J}_x, \mathbb{Q})$ corresponding to the polarization θ. Via α, the $\mathbb{Q}[\mu_m]$-action on $\mathcal{J}_x$ induces an action on V. This action satisfies

$$\mathfrak{h}_{\mathbb{Q}[\mu_m]} \neq \emptyset, \text{ and } \Psi(bv, w) = \Psi(v, b^* w),$$

for all $b \in \mathbb{Q}[\mu_m]$, all $v, w \in V$, and $Z \subset \mathrm{Sh}(\mathbb{Q}[\mu_m])$.

The isomorphism class of V^+ as a $\mathbb{Q}[\mu_m] \otimes_{\mathbb{Q}} \mathbb{C}$-module is determined by and determines the signature type $\{\mathfrak{f}(\tau) = \dim V^+_\tau\}_{\tau \in \mathcal{T}}$. By [DM86, 2.21, 2.23] (see also [Moo10, §§3.2, 3.3, 4.5]), the $H_{\mathbb{Q}[\mu_m]}(\mathbb{R})$-orbit $Y_{\mathbb{Q}[\mu_m]}$ in $\mathfrak{h}_{H_{\mathbb{Q}[\mu_m]}}$ such that

$$Z \subset \mathrm{Sh}(H_{\mathbb{Q}[\mu_m]}, Y_{\mathbb{Q}[\mu_m]})$$

corresponds to the isomorphism class of V^+ with $\mathfrak{f}$ given by (7.5). From now on, since $\mathrm{Sh}(H_{\mathbb{Q}[\mu_m]}, Y_{\mathbb{Q}[\mu_m]})$ depends only on μ_m and $\mathfrak{f}$, we denote it by $\mathrm{Sh}(\mu_m, \mathfrak{f})$.

The irreducible component of $\mathrm{Sh}(\mu_m, \mathfrak{f})$ containing Z is the largest closed, reduced and irreducible substack S of $\mathcal{A}_g$ containing Z such that the action of $\mathbb{Z}[\mu_m]$ on $\mathcal{J}$ extends to the universal abelian scheme over S. To emphasis the dependence on the monodromy datum, we denote this irreducible substack by $S(m, N, a)$.

7.5 Open questions

Suppose $g \geq 4$. Coleman conjectured that there are only finitely many smooth projective curves C of genus g such that $\mathrm{Jac}(C)$ has complex multiplication. There are special families that provide counterexamples to the Coleman conjecture for $5 \leq g \leq 7$.

If $g \geq 8$, Oort stated the expectation that there is no positive-dimensional special subvariety of $\mathcal{A}_g$ contained in the Torelli locus, with generic point contained in the open Torelli locus. Because of the André–Oort Conjecture for $\mathcal{A}_g$, Oort's expectation is equivalent to Coleman's conjecture for large g.

Here is a question that we will not address in the problem sessions.

Question 7.5.1. *What is the largest g for which there is a counterexample to the Coleman conjecture (resp. Oort's expectation?*

Chapter 8

Newton polygons for abelian varieties and curves with cyclic action

8.1 Overview

There are restrictions on the p-ranks, Newton polygons, and Ekedahl–Oort types for abelian varieties and curves having non-trivial automorphisms. This leads to open questions about whether there exist cyclic covers of curves whose Jacobians realize these invariants. Continuing the previous chapter, we consider Jacobians of curves that are cyclic covers of the projective line.

8.2 Background

8.2.1 Abelian varieties with complex multiplication

This section will be developed further at a later time.

Historically, many interesting phenomena were discovered by studying abelian varieties with complex multiplication.

For example, if m is an odd prime, then the curve $C : y^m = x(x-1)$ has genus $(m-1)/2$ and $\mathrm{Jac}(C)$ has complex multiplication by the field $\mathbb{Q}(\zeta_m)$. More generally, there are many results about quotients of Fermat curves and cyclic covers of $\mathbb{P}^1$ branched at 3 points.

The curves C provide many examples of unusual Newton polygons. Weil proved that the eigenvalues of Frobenius on $\mathrm{Jac}(C)$ can be expressed using Jacobi sums. This topic was studied by Honda, Gross-Rohrlich, Shimura-Taniyama, and Yui.

In particular, let m be odd. Let f be the order of p modulo m. If f is even and $p^f \equiv -1 \bmod m$, then $C : y^m = x(x-1)$ is supersingular. For example, the genus 6 curve $y^{13} = x(x-1)$ is supersingular if $p \not\equiv 1, 3, 9 \bmod 13$.

8.2.2 Constraints on the invariants

Consider an abelian variety X with action by the group ring $\mathbb{Z}[\mu_m]$ with signature $\mathfrak{f}$. Let $p \nmid 2m$. The interaction between the Frobenius action and the μ_m-action places constraints on the p-rank, Newton polygon, and Ekedahl–Oort type of X.

The first step of understanding those constraints is to consider the orbits o of $\times p$ on $\mathbb{Z}/m - \{0\}$. Both the Dieudonné module and the p-torsion group scheme of X decompose into pieces indexed by those orbits.

The constraints on the p-rank can be found in [Bou01]. Specifically, the maximum p-rank is bounded by the sum (over the orbits) of the length of the orbit multiplied by the minimal dimension of an eigenspace L in that orbit.

The constraints on the Newton polygon are called the Kottwitz conditions. These were developed by Kottwitz, Rapoport, and Richartz.

Definition 8.2.1. The Dieudonné module M decomposes into pieces M_o indexed by the orbits, or by the primes of $\mathbb{Q}(\zeta_m)$ above p.

The residue field of the prime acts on the piece M_o, so the multiplicity of each slope is divisible by $\#o$.

The Rosati involution $*$ acts on $\mathbb{Q}[\mu_m]$ by involution: if o is invariant under $*$ then M_o is symmetric; if not, then $M_o \oplus M_{o^*}$ symmetric.

The μ-ordinary Newton polygon μ_o for M_o has s distinct slopes where s is the number of distinct values across the orbit of $\dim(L_i)$ in the range $[1, \mathfrak{f}(i) + \mathfrak{f}(-i) - 1]$.

All Newton polygons on M_o are less ordinary than μ_o.

Definition 8.2.2. Given m and $\mathfrak{f}$, in the set of Newton polygons satisfying the Kottwitz conditions, the maximal element is called **μ-ordinary**, and the minimal element is called **basic**.

In particular, if m is prime, let f be the order of p modulo m. Then the p-rank is divisible by f.

Example 8.2.3. Moonen family M[17] Let $m = 7$, $N = 4$, and $a = (2, 4, 4, 4)$. This implies $g = 6$ and the signature is $f = (1, 2, 0, 2, 0, 1)$. Let $p \equiv 3, 5 \bmod 7$. The action of Frobenius is transitive on the eigenspaces L_i. The maximum p-rank is the stable rank of Frobenius, which is 0. The μ-ordinary Newton polygon is $G_{1,2}^2 \oplus G_{2,1}^2$; this has slopes 1/3 and 2/3, each occurring with multiplicity 6. The basic Newton polygon is supersingular.

8.3 Main theorems

Theorem 8.3.1. *Viehmann/Wedhorn: given m and $\mathfrak{f}$, each Newton polygon satisfying the Kottwitz conditions occurs on S_γ. The Newton polygon stratification of S_γ is well-understood.*

Now we can reframe the Newton polygon question for cyclic covers:

Question 8.3.2. *Let ν be a Newton polygon satisfying the Kottwitz conditions for γ with respect to p. Is there a μ_m-cover $C \to \mathbb{P}^1$ of smooth curves with monodromy datum γ such that C has Newton polygon ν?*

Here is a geometric version of this question. Consider the image T_γ° of the Torelli morphism $T : T_\gamma \to S_\gamma$.

Question 8.3.3. *Let ν be a Newton polygon satisfying the Kottwitz conditions for γ with respect to p. Does T_γ° intersect the Newton polygon stratum $S_\gamma[\nu]$?*

This question is easiest to answer for the μ-ordinary Newton polygon. Under mild conditions, Bouw proved that the maximal p-rank occurs on T_γ° [Bou01]. This result was generalized by Lin, Mantovan, and Singal in [LMS]; when $N = 4$ and $N = 5$, for all choices of m and a, they proved that the μ-ordinary Newton polygon occurs on T_γ°.

For an arbitrary large N, under certain conditions, the main result of [LMPT22] is that both the μ-ordinary and the non μ-ordinary Newton polygon occur on T_γ°.

8.4 Related results

8.4.1 Inductive results

In [LMPT22], for questions about the Newton polygon strata, we developed a method to work inductively for families of μ_m-covers as the number of branch points (and the genus) grow. The full statement of the results is too long to include here because they require some subtle conditions on the signatures.

The basic idea is that, for a fixed prime p prime with $p \nmid m$, we find inductive systems of $\gamma = (m, N, a)$ for which the open Torelli locus T_γ° intersects the μ-ordinary locus of $S[\gamma]$; and for which T_γ° intersects the non-μ-ordinary locus of $S(\gamma)$.

Here is a sample application.

Theorem 8.4.1. *[LMPT22, Theorem 1.2] Let $\gamma = (m, N, a)$ be a monodromy datum. Let p be a prime such that $p \nmid m$. Let u be the μ-ordinary Newton polygon associated to γ. Suppose there exists a μ_m-cover of $\mathbb{P}$ defined over $\overline{\mathbb{F}}_p$ with monodromy datum γ and Newton polygon u. Then, for any $n \in \mathbb{Z}_{\geq 1}$, there exists a smooth curve over $\overline{\mathbb{F}}_p$ with Newton polygon $\nu_n := u^n \oplus (0,1)^{(m-1)(n-1)}$.*

The slopes of ν_n are the slopes of u (with multiplicity scaled by n) and 0 and 1 each with multiplicity $(m-1)(n-1)$. If u is not ordinary, then for sufficiently large n, Theorem 8.4.1 demonstrates an unlikely intersection of the Newton polygon stratification and the Torelli locus in $\mathcal{A}_g$.

8.4.2 Curves that are not μ-ordinary

Consider one of the Moonen special families of cyclic covers of $\mathbb{P}^1$. In [LMPT19, Theorem 1.1] and [LMPT22, Theorem 7.1], the authors prove that every non-μ-ordinary Newton polygon ν satisfying the Kottwitz conditions occurs on the open Torelli locus of this family, for every prime p (with the condition that p is sufficiently large when ν is supersingular).

For the 14 one-dimensional Moonen special families, it is possible to say more. Building on Example 6.5.5, for 1-dim special families, there is only one option for the a-number.

Example 8.4.2. [CP, Corollary 6.4] Consider the following families of cyclic degree m covers:

$$y^m = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}.$$

For primes $p \equiv 1 \bmod m$, the number of non-ordinary curves in the family has linear rate of growth $n(p-1)$, where n is given below:

Label	m	a	g	n
$M[1]$	2	$(1,1,1,1)$	1	1/12
$M[3]$	3	$(1,1,2,2)$	2	1/6
$M[4]$	4	$(1,2,2,3)$	2	1/8
$M[5]$	6	$(2,3,3,4)$	2	1/6
$M[7]$	4	$(1,1,1,1)$	3	1/12
$M[9]$	6	$(1,3,4,4)$	3	1/12
$M[11]$	5	$(1,3,3,3)$	4	1/30
$M[12]$	6	$(1,1,1,3)$	4	1/12
$M[13]$	6	$(1,1,2,2)$	4	1/6
$M[15]$	8	$(2,4,5,5)$	5	1/8
$M[17]$	7	$(2,4,4,4)$	6	1/21
$M[18]$	10	$(3,5,6,6)$	6	3/10
$M[19]$	9	$(3,5,5,5)$	7	1/18
$M[20]$	12	$(4,6,7,7)$	7	1/6

The family $M[1]$ is the Legendre family and the families $M[3,4,5]$ are studied in [IKO86].

8.4.3 Other references

Other work on this topic can be found in [Elk11] and [Á14].

8.5 Open questions

8.5.1 Newton polygons on special abelian families

Question 8.5.1. *For one-dimensional special families of abelian (non-cyclic) covers $X \to \mathbb{P}^1$: find the Newton polygons and Ekedahl–Oort types that occur for curves in these families; for primes such that the generic curve in the family is ordinary, find the rate of growth of the number of non-ordinary curves in the family.*

8.5.2 Field of definition

Almost nothing is known about the following question.

Question 8.5.2. *Fix $g \geq 4$ and a prime p. Suppose η is a Newton polygon or Ekedahl–Oort type which occurs on $\mathcal{M}_g$ in characteristic p. Is $\mathcal{A}_g[\eta] \cap \mathcal{T}^\circ(\mathbb{F}_p)$ non-empty?*

Alternatively, does there exists a curve of type η that is defined over $\mathbb{F}_p$?

A good starting point for this question is to consider the 1-dimensional special families in Chapter 8 and consider the field of definition of the basic points.

Chapter 9

Projects

These notes are written for my project group at the 2024 Arizona Winter School. A longer more detailed version of this chapter is available upon request. If you write a paper about any of these problems, please thank the Arizona Winter School, Steven Groen, and myself.

In this chapter, we collect some of the open problems described in the lecture notes. Section 9.1 contains problems about the Torelli locus over the complex numbers. The later sections contain problems about non-ordinary Jacobians and the intersection of the open Torelli locus with the Newton polygon strata. A subset of the problems will be a focus for the AWS projects.

Most of these problems are difficult and any progress will be valuable. Sometimes I describe an open question only in a special case.

9.1 Problems in characteristic zero

Question 9.1.1. *(See Question 2.6.1) [ES93] Given $g \geq 2$, does there exist a smooth curve X of genus g such that the Jacobian J_X is isogenous to a product of g elliptic curves?*

Currently, the first unknown cases are $g = 59$ and $g = 66$.

Question 9.1.2. *(See Question 5.5.1) If $g \geq 3$, what is the maximum dimension of a complete subspace of $\mathcal{M}_g$?*

Currently, it is not known whether there is a complete subspace of dimension 2 in $\mathcal{M}_4$.

Question 9.1.3. *(See Question 7.5.1) What is the largest g for which there is a counterexample to the Coleman conjecture? This means that there are infinitely many smooth projective curves C of genus g such that* $\mathrm{Jac}(C)$ *has complex multiplication.*

Is Oort's conjecture true? It states, if $g \geq 8$, that there is no positive-dimensional special subvariety of $\mathcal{A}_g$ contained in the Torelli locus, and intersecting the open Torelli locus.

Currently, I believe the situation is determined for families of curves with automorphisms for $g \leq 9$, with all counter examples occurring for $g \leq 7$. There are a lot of references on this topic. For a starting point, see the work of Moonen [Moo10], Moonen and Oort [MO13], or the work of Frediani, Ghigi, and Penegini [FGP15], which contains many references.

9.2 Counting non-ordinary curves

The idea in this section is to count the number of curves having a particular arithmetic invariant η. Using the main ideas in arithmetic statistics, this count will provide information about the dimension of the stratum S_η of curves with that invariant and the number of components of S_η. These questions will be fun from a computational standpoint. It is not clear to me how much data is needed to provide good evidence.

9.2.1 The question

Question 9.2.1. *(See Question 4.5.2) Determine the rate of growth of the number of curves over $\mathbb{F}_p$ (up to geometric isomorphism) having the following types as p grows.*

1. *Non-ordinary curves of genus 4 (resp. of genus 5);*
2. *p-rank 0 curves of genus 4 (resp. of genus 5);*
3. *Supersingular curves of genus 4.*

See also Question 6.5.1 and Question 6.5.3. Let's work over the finite field $K = \mathbb{F}_p$ of odd characteristic p. Several papers of Harashita and Kudo may be helpful for these questions.

9.3 Supersingular curves in special families

This section contains a series of problems about supersingular curves in special families of curves. We consider a family F of curves that are Galois covers $C \to \mathbb{P}^1$. Recall that the family is *special* if the image of the Torelli morphism is open and dense in a component of the associated Shimura variety. Intuitively speaking, this means that the dimension of the family of curves equals the dimension of the family of abelian varieties whose endomorphism algebra has a compatible structure.

This section is organized into three subsections that describe different types of families. Based on the state of knowledge, we focus on a different question in each subsection.

9.3.1 Supersingular curves in two-dimensional special families

Question 9.3.1. *The following result was proven in [LMPT22, Theorem 7.1] for primes (satisfying the given congruence condition) that are sufficiently large. In the family F, for the prime $p >> 0$, there is a smooth curve that is supersingular.*

1. *$M[6]$: the family is $F : y^3 = x(x-1)(x-t_1)(x-t_2)$, so $g = 3$, with $p \equiv 2 \bmod 3$.*
2. *$M[8]$: the family is $F : y^4 = x(x-1)(x-t_1)^2(x-t_2)^2$, so $g = 3$, with $p \equiv 3 \bmod 4$.*
3. *$M[10]$: the family is $F : y^3 = x(x-1)(x-t_1)(x-t_2)(x-t_3)$, so $g = 4$, with $p \equiv 2 \bmod 3$.*
4. *$M[14]$: the family is $F : y^6 = x^2(x-1)^2(x-t_1)^2(x-t_2)^3$, so $g = 4$, with $p \equiv 5 \bmod 6$.*
5. *$M[16]$: the family is $F : y^5 = x(x-1)(x-t_1)(x-t_2)$, so $g = 6$, with $p \equiv 2, 3, 4 \bmod 5$.*

Give a complete description of the supersingular locus in these families. In particular, remove the condition that the prime needs to be sufficiently large.

The case $M[16]$ is the most interesting one in the table above.

9.3.2 Field of definition of supersingular curves in special families

Question 9.3.2. *The following result was proven for primes (satisfying the given congruence condition) that are sufficiently large [LMPT19, Theorem 7.1]: In the family F, for the prime $p >> 0$, there is a smooth curve that is supersingular.*

1. $M[15] : y^8 = x^2(x-1)(x-t)$, *with genus* 5, *when* $p \equiv 7 \bmod 8$*;*
2. $M[17] : y^7 = x(x-1)(x-t)$, *with genus* 6, *when* $p \equiv 3, 5, 6 \bmod 7$*;*
3. $M[19] : y^9 = x(x-1)(x-t)$, *with genus* 7, *when* $p \equiv 2 \bmod 3$*;*
4. $M[20] : y^{12} = x^4(x-1)(x-t)$, *with genus* 7, *when* $p \equiv 11 \bmod 12$.

What is the field of definition of those supersingular curves?

Remark 9.3.3. A result that removes the condition $p >> 0$ may appear soon.

9.3.3 Special families of non-cyclic covers

Question 9.3.4. *(See Question 8.5.1) In [MO13, Table 2, page 38], Moonen and Oort found seven special families of curves, for which each curve in the family is an abelian (non-cyclic) cover $C \to \mathbb{P}^1$. We focus on the five families for which the genus is bigger than* 2 *(namely,* 3 *or* 4*). For each of the families, for a fixed prime p:*

1. *Find the Newton polygons satisfying the Kottwitz conditions for the family.*
2. *Under what conditions is there a smooth supersingular curve in the family? Over what field is it defined?*
3. *For the four such families that are* 1*-dimensional, find the rate of growth of the number of supersingular curves in the family.*

Question 9.3.5. *In [FGP15, Table 2], Frediani, Ghigi, and Penegini found thirteen special families of curves for which each curve in the family is a non-abelian Galois cover $C \to \mathbb{P}^1$. We focus on the ten families for which the genus is bigger than* 2 *(namely,* 3, 4, 5, *or* 7*). For each of the families, for a fixed prime p:*

1. *Find the Newton polygons satisfying the Kottwitz conditions for the family.*
2. *Under what conditions is there a smooth supersingular curve in the family? Over what field is it defined?*
3. *For the eight such families that are* 1*-dimensional, find the rate of growth of the number of supersingular curves in the family.*

9.4 Double covers

Let p be an odd prime. Let $k = \bar{k}$ with $\mathrm{char}(k) = p$.

9.4.1 The p-ranks of double covers of an elliptic curve

Let E be an elliptic curve. Let $n \geq 1$. Let B be a set of $2n$ distinct points of E. We are going to study double covers $\phi : C \to E$ branched at B. Try to prove this lemma!

The involution on C acts as an automorphism of order 2 on $\mathrm{Jac}(C)$. The new part $\mathrm{Jac}_{new}(C)$ is the subabelian variety of $\mathrm{Jac}(C)$ which is negated under this action.

Recall that a curve is ordinary if its p-rank equals its genus.

Question 9.4.1. *Prove that* $\mathrm{Jac}_{new}(C)$ *is ordinary for a generic choice of* $2n$ *points.*

Prove that there exists a set of $2n$ *points such that* $\mathrm{Jac}_{new}(C)$ *is not ordinary.*

Question 9.4.2. *(See Question 6.5.7) Study the dimensions of the p-rank strata of the moduli space of double covers of a fixed elliptic curve with* $2n$ *branch points.*

9.4.2 Non-ordinary curves in complete families of $\mathcal{M}_g$

Recall the construction of a complete curve $\mathcal{W}$ in $\mathcal{M}_g$ by Gonzalez Diez and Harvey.

Theorem 9.4.3. *[GDH91] If* $g \geq 3$*, there exists a complete curve in* $\mathcal{M}_g$*.*

Proof. Construction: Take $E : y^2 = x^3 - 1$ an elliptic curve and $X : y^2 = x^6 - 1$ which has genus 2. The double cover $\tau : X \to E$ is branched above $(0, i)$ and $(0, -i)$. Let r be even. Choose points $Q_1 = 0_E, Q_2, \ldots, Q_r \in E$ such that $Q_i - Q_j$ is not a 2-torsion point. Let $W = \{(P, P +_E Q_2, \ldots, P +_E Q_r) \mid P \in E\}$. Note that $W \subset E^r - \Delta$ and $W \cong E$. Let $T \subset X^r - \Delta$ be the set of points $\vec{x} = (x_1, \ldots, x_r)$ such that $\tau(x_i) = \tau(x_1) +_E Q_i$. Then T is complete and $\dim(T) \geq 1$.

Now take $r = 2(g-3)$. For each point $\vec{x} \in T$, consider the cover $Z \to X$ branched at the r coordinates of $\vec{x}$. By the Riemann–Hurwitz formula, Z has genus g. The curves are not isomorphic (by Riemann's existence theorem). This produces a complete curve in $\mathcal{M}_g$. □

Question 9.4.4. *Let* C *be a curve produced in this construction. What are the possibilities for the Newton polygon of* C*?*

9.5 The geometry of the supersingular locus

The questions here are very interesting to me, but I do not have a strategy to solve them.

Question 9.5.1. *(See Question 6.5.6) Determine the intersection of the supersingular locus of $\mathcal{M}_3$ with the boundary of $\mathcal{M}_3$; similar question for the hyperelliptic locus $\mathcal{H}_3$.*

Here is more information about this question.

Let $\mathcal{A}_g[ss]$ (resp. $\mathcal{M}_g[ss]$) denote the supersingular locus of $\mathcal{A}_g$ (resp. $\mathcal{M}_g$).

The dimension of each component of $\mathcal{A}_3[ss]$ is two. As seen in Valentijn's lecture series, the generic point of each component represents an abelian variety X having p-rank 0 and a-number 1. This implies that X is not isomorphic to a product of p.p. abelian varieties of smaller dimension. Thus X is the Jacobian of a smooth curve of genus 3. Since the Torelli map is injective, it follows that $\mathcal{M}_3[ss]$ is non-empty and its components have dimension 2. Let S denote one such component.

By Theorem 5.3.1 ([Dia87a, Theorem 4], [Loo95b, page 412]), if $Z \subset \mathcal{M}_3$ is complete, then $\dim(Z) \leq 1$. Thus S is not complete in $\mathcal{M}_3$. Let $\bar{S}$ denote its closure in $\bar{\mathcal{M}}_3$. Because S is contained in the p-rank 0 locus, $\bar{S}$ does not intersect Δ_0. Thus $\bar{S}$ intersects Δ_1 and the intersection $\bar{S} \cap \Delta_1$ has dimension 1.

However, there are components of $\bar{\mathcal{M}}_3[ss]$ that have dimension 2 and are completely contained in the boundary of $\bar{\mathcal{M}}_3$. Specifically, these are the components of the image of $\mathcal{M}_{1;1}[ss] \times \bar{\mathcal{M}}_{2;1}[ss]$ under the clutching map. These components have dimension 2. Concretely, we take a supersingular elliptic curve E, marked at the identity point. There is a 1-parameter family of supersingular curves C of genus 2, and a 1-dimensional choice of marked point P on C. Then we clutch C and E together at the marked points.

The question asks for a description of the intersection of $\bar{S}$ with Δ_1. Specifically, we would like to understand which points of $\mathcal{M}_{1;1}[ss] \times \bar{\mathcal{M}}_{2;1}$ are in the intersection.

Here is an alternative way to phrase this question.

Question 9.5.2. *Which points of $\mathcal{M}_{1;1}[ss] \times \bar{\mathcal{M}}_{2;1}$ can be deformed to a smooth curve of genus 3 that is supersingular?*

There is also a hyperelliptic version of this question. In that case, a component S of $\mathcal{H}_3[ss]$ has dimension 1. The hyperelliptic locus is affine so S meets the boundary of $\mathcal{H}_3$, specifically Δ_1. The intersection of $\bar{S}$ with Δ_1 has dimension 0. However, there are components of $\bar{\mathcal{H}}_3[ss]$ that have dimension 1 and are fully contained in the boundary. Specifically, these are in the image of $\mathcal{H}_{1;1} \times \mathcal{H}_{2;1}$. This has dimension 1 because the marked points need to be ramification points for the hyperelliptic involution. So the question is which supersingular curves of genus 2 can appear in this intersection.

These questions are for $g = 3$, which is easier because the open Torelli locus is open and dense in $\mathcal{A}_3$, meaning that almost every p.p. abelian threefold is the Jacobian of a smooth curve of genus 3. (These questions can be generalized for every $g \geq 3$.) However, one reason they are difficult is the following. For a curve C of genus 3 over $\mathbb{F}_q$, the supersingular condition can be described using the Newton polygon of its L-polynomial. However, to answer these questions, I think it is necessary to have an algebraic description of the supersingular locus, and this description may be highly dependent on the prime p.

9.6 Problem about Ekedahl–Oort strata

This project is still in development. Based on feedback from the project group, this section will not be developed further at this time.

Here is a question from Chapter 3. The main technique to approach this problem uses algebra and combinatorics. It has a potential geometric application about Ekedahl–Oort types of Jacobians of curves.

Question 9.6.1. *(See Question 3.5.1) For* $5 \leq g \leq 10$*, determine the Newton polygons (resp. Ekedahl–Oort types) having* p*-rank* 0 *with this property:*

1. *in the partial ordering of Newton polygons (resp. Ekedahl–Oort types) of* $\mathcal{A}_g$*, the distance to the ordinary type is at most* $2g-2$*.*

In other words, determine the Newton polygons and Ekedahl–Oort types having p-rank 0 whose strata have codimension at most $2g-2$ in $\mathcal{A}_g$.

Here is some motivation for this question. Try to prove these lemmas!

Lemma 9.6.2. *Suppose* $S \subset \mathcal{A}_g$ *is such that* $\mathrm{codim}(S, \mathcal{A}_g) \leq 2g-2$*. If* S *intersects the image of* $\mathcal{M}_g^{ct}$ *in* $\mathcal{A}_g$*, then the intersection has dimension at least* $g-1$*.*

Example 9.6.3. Suppose $g = 4$. We are looking for Newton polygons and Ekedahl–Oort types that have p-rank 0 and whose strata have codimension at most 6 in $\mathcal{A}_4$. This means the strata has dimension at least 4. For Newton polygons, the only option is the supersingular one. For Ekedahl–Oort types, the options are:

1. $[0, 1, 2, 3]$ stratum has dimension 6.
2. $[0, 1, 2, 2]$ stratum has dimension 5.
3. $[0, 1, 1, 2]$ stratum has dimension 4.

For each of these: can you tell whether the stratum intersects the image of $\mathcal{M}_g^{ct}$ in $\mathcal{A}_4$? If yes, what can you say about the intersection?

For working on this question, it will be important to understand how to describe the structure of the mod p Dieudonné module for the Ekedahl–Oort type.

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