ALGEBRAIC CYCLES
ON ABELIAN VARIETIES

Projects for the Arizona Winter School 2024

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Preface

In this text, I describe some projects that connect to my AWS lectures about algebraic cycles on abelian varieties.

The first project is about integral aspects of Fourier duality and the Beauville decomposition. The key idea is to use the results of Pappas [14] about integral Grothendieck–Riemann–Roch to control the denominators that are needed. This leads to some results that, to my knowledge, are not yet in the literature.

In the second project, the idea is to learn about Chow motives, and to see how the language of motives can help to prove and interpret results about Chow groups. In particular, we want to discuss Chow motives of non-simple abelian varieties. As an application, we obtain very nice results about Chow motives (and hence also Chow groups) of supersingular abelian varieties that are due to Fakhruddin [6] on the level of Chow groups and to Fu–Li [8] on the level of Chow motives. This project is aimed at students who have not worked with Chow motives before. The results about supersingular abelian varieties are known, but I believe it should be possible to extend some of these ideas to the study of $\mathrm{CH}(E^g)$, where E is any CM elliptic curve.

In the third project we consider a curve C of genus $g \geq 2$, which we embed into its Jacobian J using a 0-cycle ξ on C of degree 1. Then we study the relation between the modified diagonal classes of C that were first introduced by Gross and Schoen, and the vanishing of the components $[C]_{(s)}$ in the Beauville decomposition of $\mathrm{CH}(J)$. If $\xi = x_0$ is a base point on C then the vanishing of $[C]_{(s)}$ for some $s \geq 1$ is equivalent to the vanishing of the modified diagonal $\Gamma^{s+2}(C, x_0)$. However, this setup is too restrictive; for instance, it is known that $\Gamma^3(C, x_0)$ can only vanish if $(2g-2) \cdot x_0$ is a canonical divisor, and in general there are no points x_0 with this property. It is therefore natural to allow ξ to be an arbitrary 0-cycle of degree 1; but in that case many basic properties still need to be worked out. For instance, there are at least three natural ways to generalize the Gross–Schoen modified diagonal classes, and it is unclear which of the three has the best properties. There is really a lot to explore here.

These projects are intended as a starting point for possible discussions. If such discussions branch out in other directions, that's perfectly fine; there is a lot of interesting material available, and it would be no problem if things diverge from the topics in these notes. I look forward to working with the participants of the AWS!

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1. Integral aspects of Fourier duality and the Beauville decomposition

1.1. Let X be a g-dimensional abelian variety over a field k, with dual abelian variety X^t. Let $\mathscr{P}$ be the Poincaré line bundle on $X \times X^t$, and write $\wp = c_1(\mathscr{P}) \in \mathrm{CH}^1(X \times X^t)$ for the corresponding class. As discussed in Section 8 of the Lecture Notes, the correspondence

$$\mathrm{ch}(\mathscr{P}) = \exp(\wp) = 1 + \wp + \tfrac{\wp^2}{2!} + \tfrac{\wp^3}{3!} + \cdots + \tfrac{\wp^{2g}}{(2g)!} \qquad \in \mathrm{CH}(X \times X^t)_{\mathbb{Q}}$$

defines the Fourier transform $\mathscr{F}_X = \mathrm{ch}(\mathscr{P})_*$. Concretely, this means that $\mathscr{F}_X \colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X^t)_{\mathbb{Q}}$ is defined by $\mathscr{F}_X(\alpha) = \mathrm{pr}_{X^t,*}\bigl(\mathrm{pr}_X^*(\alpha) \cdot \mathrm{ch}(\mathscr{P})\bigr)$. The basic properties of this are:

(1) $\mathscr{F}_{X^t} \circ \mathscr{F}_X = (-1)^g \cdot [-1]_X^*$, and hence $\mathscr{F}_X$ is an isomorphism;

(2) for x, $y \in \mathrm{CH}(X)_{\mathbb{Q}}$ we have the relations

$$\mathscr{F}_X(x \star y) = \mathscr{F}_X(x) \cdot \mathscr{F}_X(y)\,, \qquad \mathscr{F}_X(x \cdot y) = (-1)^g \cdot \mathscr{F}_X(x) \star \mathscr{F}_X(y)\,.$$

Closely related to this is that we have Beauville's decomposition

$$\mathrm{CH}^i(X)_{\mathbb{Q}} = \bigoplus_s \mathrm{CH}^i_{(s)}(X)_{\mathbb{Q}}$$

which can be characterised by its property that (for n an integer) $[n]_X^*$ is multiplication by n^{2i-s} on the summand $\mathrm{CH}^i_{(s)}(X)_{\mathbb{Q}}$. The relation to Fourier duality is that

$$\mathrm{CH}^i_{(s)}(X)_{\mathbb{Q}} = \bigl\{\alpha \in \mathrm{CH}^i(X)_{\mathbb{Q}} \bigm| \mathscr{F}_X(\alpha) \in \mathrm{CH}^{g-i+s}(X^t)\bigr\}\,.$$

The leading questions in this project are:

- Can we define a Fourier transform with integral coefficients, rather than $\mathbb{Q}$-coefficients? Or can we at least bound the denominators that are needed?
- Does there exist an integral version of Beauville's decomposition, or a version in which we can control the denominators that are needed?

1.2. Suppose we wanted to define an *integral* version of Fourier duality on the abelian variety X. We are then confronted with two issues:

- The Fourier transform $\mathscr{F}_X$ is defined by the correspondence $\mathrm{ch}(\mathscr{P}) \in \mathrm{CH}(X \times X^t)_{\mathbb{Q}}$, and a priori we need denominators to define this class.
- In the proof of the above properties (1) and (2) we may need $\mathbb{Q}$-coefficients; these properties may not hold integrally.

If M and N are positive integers, let us say that we have an (M,N)-integral Fourier duality on X if we have *integral* transformations

$$\mathsf{F} \colon \mathrm{CH}(X) \to \mathrm{CH}(X^t)\,, \qquad \mathsf{F}^t \colon \mathrm{CH}(X^t) \to \mathrm{CH}(X)\,,$$

with the following properties:

(0) the induced maps $\mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X^t)_{\mathbb{Q}}$ and $\mathrm{CH}(X^t)_{\mathbb{Q}} \to \mathrm{CH}(X)_{\mathbb{Q}}$ equal $M \cdot \mathscr{F}_X$, respectively $M \cdot \mathscr{F}_{X^t}$;

(1) $N \cdot (\mathsf{F}^t \circ \mathsf{F}) = M^2 N \cdot (-1)^g \cdot [-1]_X^*$;

(2) for x, $y \in \mathrm{CH}(X)$ we have the relation

$$M \cdot \mathsf{F}(x \star y) = \mathsf{F}(x) \cdot \mathsf{F}(y)\,.$$

In other words: the operators F lift the operators $M \cdot \mathscr{F}$, and the expected duality relation only hold after multiplication by an extra factor of N.

Remarks. (1) If $(\mathsf{F}, \mathsf{F}^t)$ is an (M,N)-integral Fourier duality then $(N \cdot \mathsf{F}, N \cdot \mathsf{F}^t)$ is an $(MN,1)$-integral Fourier duality. One could therefore say that the challenge is to find an (M,N)-integral Fourier duality with M as small as possible.

(2) In [2], Proposition 3′, it is shown that, for a given X/k, there exists *some* integer M such that there exists an $(M,1)$-integral Fourier duality. However, the proof is not effective and does not give a concrete value of M.

(3) You may wonder why we only require the relation $M \cdot \mathsf{F}(x \star y) = \mathsf{F}(x) \cdot \mathsf{F}(y)$ to hold and not also the second identity in 1.1(2). A version of this second identity can be deduced from the first, but the required constants are not as sharp; see the next exercise. Note that when we work with $\mathbb{Q}$-coefficients, the relation $\mathscr{F}(x \star y) = \mathscr{F}(x) \cdot \mathscr{F}(y)$ has a direct proof, and then the other identity is obtained using duality.

Exercise. Assume $(\mathsf{F}, \mathsf{F}^t)$ is an (M,N)-integral Fourier duality, as above. Show that for all x, $y \in \mathrm{CH}(X)$ we have the relation $N^2 M^4 \cdot \mathsf{F}(x \cdot y) = N^2 M^3 \cdot (-1)^g \cdot \mathsf{F}(x) \star \mathsf{F}(y)$.

1.3. Define $\gamma \in \mathrm{CH}(X \times X^t)$ by

$$\gamma = (2g)! \cdot \mathrm{ch}(\mathscr{P}) = \sum_{i=0}^{2g} \tfrac{(2g)!}{i!} \cdot \wp^i\,,$$

and define $\mathsf{F} = \gamma_* \colon \mathrm{CH}(X) \to \mathrm{CH}(X^t)$. Define $\mathsf{F}^t \colon \mathrm{CH}(X^t) \to \mathrm{CH}(X)$ analogously. Note that the induced map $\mathsf{F}_{\mathbb{Q}} \colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X^t)_{\mathbb{Q}}$ is $(2g)! \cdot \mathscr{F}_X$, and likewise for $\mathsf{F}^t_{\mathbb{Q}}$. The goal is now to determine an explicit integer N such that the pair $(\mathsf{F}, \mathsf{F}^t)$ is an (M,N)-integral Fourier transform, where from now on we set $M = (2g)!$. The key idea is to use the results of Pappas [14].

- Carefully go through the proof of Theorem 8.3 in the Lecture Notes. We want to calculate $\mathsf{F}^t \circ \mathsf{F}$ using the same strategy. Convince yourself that the key issue is to calculate (with integral coefficients!) the class $\mathrm{pr}_{1,*}(\gamma)$, where $\mathrm{pr}_1 \colon X \times X^t \to X$ is the projection map.
- Work through the first couple of pages of the paper [14]. Note that the main result of [14] is only proven over fields k of characteristic 0; we therefore make this assumption on the base field. Write down precisely what the result of Pappas gives in the situation of a projective abelian scheme $X \to S$ of relative dimension g. A key point here is that the relative tangent sheaf $T_{X/S}$ is a trivial vector bundle, so that $c_i(T_{X/S}) = 0$ for all $i \geq 1$.

- Let X/k be a g-dimensional abelian variety. Let $e\colon \mathrm{Spec}(k) \to X$ be the inclusion of the origin. Of course, $\mathrm{CH}\bigl(\mathrm{Spec}(k)\bigr) = \mathrm{CH}^0\bigl(\mathrm{Spec}(k)\bigr) = \mathbb{Z}\cdot\bigl[\mathrm{Spec}(k)\bigr]$. Define integers T_m as in [14], formula (1.2), and let $\mathfrak{s}_n = n!\cdot \mathrm{ch}_n$ as in [14], Section 2.b. Use the results of Pappas to show that

$$\tfrac{T_n}{n!}\cdot \mathfrak{s}_n\bigl(e_*[\mathscr{O}_{\mathrm{Spec}}(k)]\bigr) = \begin{cases} T_g\cdot e_*[\mathrm{Spec}(k)] & \text{if } n = g,\\ 0 & \text{otherwise.}\end{cases}$$

- Prove that we can take $N = T_{2g}/(2g)!$, which by [14], Lemma 2.1 is an integer. Can you further sharpen the value of N that is needed?

1.4. Next we should also like to obtain an "integral" version of the Beauville decomposition. Note that with $\mathbb{Q}$-coefficients, the existence of a Beauville decomposition is not a formal consequence of Fourier duality. A key point is that we need Lemma 8.7(2) in the Lecture Notes.

Let X/k be a g-dimensional abelian variety, where we assume $\mathrm{char}(k) = 0$. If you have found an (M, N)-integral Fourier duality with $M = (2g)!$, you should also be able to get a Beauville decomposition of $\mathrm{CH}(X)\otimes\Lambda$, for some coefficient ring Λ of the form $\Lambda = \mathbb{Z}[1/MN]$.

Write $\mathrm{CH}(X)_\Lambda = \mathrm{CH}(X)\otimes\Lambda$. The main point in this last part of the project is that you should find a definition of subspaces $\mathrm{CH}^i_{(s)}(X)_\Lambda \subset \mathrm{CH}(X)_\Lambda$ that works. In any case, you need that some version of [3], Proposition 1 holds, and such that also the analogue of Lemma 8.7(2) in the Lecture Notes holds. (If you have found the optimal value of N then you should find that in fact we can take $\Lambda = \mathbb{Z}[1/(2g+1)!]$.)

1.5. As a final topic, one could try to show that the $\mathfrak{sl}_2$-action that is discussed in Section 9 of the Lecture Notes works with coefficients in Λ.

2. Chow motives of abelian varieties

2.1. This project is aimed at students who have not worked with Chow motives before. If you want to work on this, you are advised to first read Part 3 of the Lecture Notes, or some other introduction to Chow motives. Once you have a basic understanding of the definitions, you should try to start working on the questions below, and along the way you will start seeing how powerful the theory of Chow motives really is. As we shall see, the tensor structure on the category $\mathsf{CHM}(k)$ of Chow motives plays a main role.

2.2. In what follows we use multi-index notation: elements of $\mathbb{Z}^t$ are written as $\underline{n} = (n_1, \ldots, n_t)$, and we define $|\underline{n}| = n_1 + \cdots + n_t$. If $\underline{i} = (i_1, \ldots, i_t)$ is another t-tuple then we define $\underline{n}^{\underline{i}} = n_1^{i_1} \cdots n_t^{i_t}$.

We work over some base field k. Let X be an abelian variety over k which decomposes as a product, say $X = X_1 \times \cdots \times X_t$. Let $g_j = \dim(X_j)$ and $\underline{g} = (g_1, \ldots, g_t)$, so that $\dim(X) = |\underline{g}|$. For $\underline{n} \in \mathbb{Z}^t$ we have an endomorphism $[\underline{n}]_X \colon X \to X$, given by $(x_1, \ldots, x_t) \mapsto (n_1x_1, \ldots, n_tx_t)$. The usual multiplication-by-n maps are the special case where $n_1 = \cdots = n_t = n$.

A first question we ask is whether the usual Beauville decomposition can be refined as a decomposition into simultaneous "eigenspaces" with respect to all operators $[\underline{n}]_X^*$. While it seems plausible that this should work, there is a risk that the problem becomes notationally rather involved. The theory of Chow motives (of which we shall only need a couple of basic principles) can help us to solve the problem is a very elegant way.

- Have a look at the Deninger–Murre theorem, which is Theorem 13.1 in the Lecture Notes. Do you see what kind of "multi-factor" analogue we would expect in the above situation of a product abelian variety?
- Use the tensor structure on the category (see Basic Fact 12.4(3) in the Lecture Notes) to obtain a decomposition of $\mathfrak{h}(X)$ from the Deninger–Murre decompositions of the factors X_j. (It's really that simple!)
- If $f \colon X \to Y$ is an isogeny of abelian varieties, show that the induced morphism of Chow motives $f^* \colon \mathfrak{h}(Y) \to \mathfrak{h}(X)$ is an isomorphism. Using this, we can extend the result to arbitrary non-simple abelian varieties, even those that are not a product of smaller factors.

2.3. We now want to use the tensor structure on the category $\mathsf{CHM}(k)$ to obtain a very nice description of the Chow motive of a supersingular abelian variety. In what follows, we assume that k is an algebraically closed field of characteristic $p > 0$. Let X be a g-dimensional abelian variety over k, where we assume that $g \geq 2$. (The case $g = 1$ is of little interest.)

There are several ways to define when X/k is said to be *supersingular*. We refer to Section 1.3 of the Lecture Notes of V. Karemaker or Chapter 3 of the Lecture Notes of R. Pries for further discussion. For this project, all you really need to know are the following facts.

2.4 Facts. (1) If p is a prime number, there exists a supersingular elliptic curve E over $\mathbb{F}_p$. With $k = \bar{k}$ of characteristic p as above, every supersingular elliptic curve over k is defined over the subfield $\mathbb{F}_{p^2} \subset k$, and any two supersingular elliptic curves over k are isogenous.

(2) If E/k is a supersingular elliptic curve, $\mathrm{End}^0(E)$ is a quaternion algebra with centre $\mathbb{Q}$. (More precisely, it is the unique such quaternion algebra that is ramified at the primes p and ∞, and which splits at all other primes.)

(3) Let $k = \bar{k}$ be as above, and let g be an integer with $g \geq 2$. If $E_1, \ldots, E_g, E'_1, \ldots, E'_g$ are arbitrary supersingular elliptic curves over k, we have $E_1 \times \cdots \times E_g \cong E'_1 \times \cdots \times E'_g$. (This is due to Deligne.)

(4) Let E/k be any supersingular elliptic curve over k (which exists by the first fact). If X/k is a g-dimensional supersingular abelian variety with $g \geq 2$ then X is isogenous to E^g. (Caution: it is not true in general that X can be defined over a finite field.)

2.5. To simplify notation, we now fix the base field $k = \bar{k}$, and we simply write E instead of E_k, where of course we assume E to be supersingular. As noted above, isogenous abelian varieties have isomorphic Chow motives. By the facts just stated, we may from now on assume that $X = E^g$.

Our goal is to find a description of the Chow motive of X, using the relation $\mathfrak{h}(X) = \mathfrak{h}(E)^{\otimes g} = [\mathbf{1} \oplus \mathfrak{h}^1(E) \oplus \mathbf{1}(-1)]^{\otimes g}$. The key point is to understand the motive $\mathfrak{h}^1(E) \otimes \mathfrak{h}^1(E)$. We have the dual motive $\mathfrak{h}^1(E)^\vee$, which satisfies $\mathfrak{h}^1(E)^\vee \cong \mathfrak{h}^1(E)\big(1\big)$, so it suffices to better understand the motive

$$\underline{\mathrm{End}}\big(\mathfrak{h}^1(E)\big) := \mathfrak{h}^1(E)^\vee \otimes \mathfrak{h}^1(E)\,.$$

(This is an algebra in the category $\mathsf{CHM}(k)$.) The advantage of this change in perspective is that we know that

$$\mathrm{Hom}_{\mathsf{CHM}(k)}\big(\mathbf{1}, \underline{\mathrm{End}}(\mathfrak{h}^1(E))\big) = \mathrm{End}_{\mathsf{CHM}(k)}(\mathfrak{h}^1(E)) \cong \mathrm{End}^0(E)\,.$$

(For the last isomorphism, see Remark 13.4 in the Lecture Notes.) This gives us a morphism of motives

$$\mathrm{End}^0(E) \otimes \mathbf{1} \longrightarrow \underline{\mathrm{End}}\big(\mathfrak{h}^1(E)\big)\,. \tag{2.5.1}$$

(The source is just a direct sum of 4 copies of the unit motive $\mathbf{1}$.) The main steps to work through now are the following; here we let $b_i = \binom{2g}{i}$ be the Betti number in degree i of X.

- Show, using the main result of [1], that the morphism (2.5.1) is an isomorphism. (You may need some help with this.)
- Derive from this that $\mathfrak{h}^1(E)^{\otimes 2} \cong \mathbf{1}(-1)^{\oplus 4}$. (Note: this is very special for supersingular elliptic curves; it certainly does not hold for arbitrary elliptic curves.)
- Show that

$$\mathfrak{h}^{2n}(X) \cong \mathbf{1}(-n)^{\oplus\, b_{2n}} \tag{2.5.2}$$

for all $n \geq 1$ and that

$$\mathfrak{h}^{2n+1}(X) \cong \mathfrak{h}^1(E)\big(-n\big)^{\oplus\, b_{2n+1}/2}$$

for all $n \geq 0$. (You may use the combinatorial identity $\sum_{a+2b=m} \binom{g}{a\,b} \cdot 2^a = \binom{2g}{m}$, where $\binom{g}{a\,b} = g!/a!\,b!\,(g-a-b)!$. Both sides compute the coefficient of x^m in $(1+2x+x^2)^g = (1+x)^{2g}$.)

- Passing to Chow groups, deduce from these results that for a supersingular abelian variety X/k we have

$$\mathrm{CH}^j(X) = \mathrm{CH}^j_{(0)}(X) \oplus \mathrm{CH}^j_{(1)}(X) \qquad \text{for all } j \geq 0,$$

with

$$\mathrm{CH}^j_{(0)}(X) \cong \mathbb{Q}^{b_{2j}}\,, \qquad \mathrm{CH}^j_{(1)}(X) \cong \big(E(k) \otimes \mathbb{Q}\big)^{\oplus\, b_{2j-1}/2}\,.$$

Thus, there are only two 'layers' in the Beauville decomposition that occur. This shows that Theorem 11.7 in the Lecture Notes (due to Bloch) does not hold over fields of characteristic p.

Note: These last results about the Chow groups of X are due to Fakhruddin [6]; the results about the structure of the motives $\mathfrak{h}^i(X)$ were proven by Fu and Li in [8], though their proof that the morphism (2.5.1) is an isomorphism is inadequate and the above method gives a much simpler calculation.

2.6. These results about supersingular abelian varieties hinge on two facts: (a) if X is a g-dimensional supersingular abelian variety, $X \sim E^g$ for a supersingular elliptic curve E; (b) if E is a supersingular curve, the motive $\mathfrak{h}^1(E)\otimes\mathfrak{h}^1(E)$ is purely of Tate type, more precisely, it is isomorphic to $\mathbf{1}(-1)^{\oplus 4}$. Both facts are very particular for supersingular abelian varieties. However, it should still be possible to fully describe the Chow motive and the Chow groups of E^g if E is an elliptic curve (over an algebraically closed field k, say) whose endomorphism algebra $K = \mathrm{End}^0(E)$ is an imaginary quadratic field. General results about the motivic decomposition of abelian varieties with a nontrivial endomorphism algebra have been given in [11], but here we may be able to give a more explicit description.

Let then E be an elliptic curve such that $K = \mathrm{End}^0(E)$ is an imaginary quadratic field. To begin with, we of course have

$$\mathfrak{h}(E) = \mathfrak{h}^0(E) \oplus \mathfrak{h}^1(E) \oplus \mathfrak{h}^2(E)\,,$$

with $\mathfrak{h}^0(E) \cong \mathbf{1}$ and $\mathfrak{h}^2(E) \cong \mathbf{1}(-1)$. We have $\mathrm{CH}\big(\mathfrak{h}^1(E)\big) = \mathrm{CH}^1\big(\mathfrak{h}^1(E)\big)$, and this group is nonzero, unless k is the algebraic closure of a finite field. To understand $\mathfrak{h}(E^g) = \mathfrak{h}(E)^{\otimes g}$, it suffices to understand the motives $\mathfrak{h}^1(E)^{\otimes m}$ for $m \geq 2$. Note that $\mathfrak{h}^1(E)$ is a K-module in the category $\mathsf{CHM}(k)$ (i.e., a motive with a given action of K by endomorphisms). If M is a K-module in $\mathsf{CHM}(k)$ then we have objects such as $M \otimes_K M$, and also we can take symmetric and exterior powers relative to K. If we take $g = 2$, I believe it is the case that

$$\mathfrak{h}^1(E) \otimes \mathfrak{h}^1(E) \cong \mathbf{1}(-1)^{\oplus 2} \oplus \big(\wedge^2_K \mathfrak{h}^1(E)\big)\,,$$

and $\mathrm{CH}\big(\wedge^2_K \mathfrak{h}^1(E)\big) = \mathrm{CH}^2\big(\wedge^2_K \mathfrak{h}^1(E)\big)$. (The summand $\mathbf{1}(-1)^{\oplus 2}$ comes from the endomorphisms of E.) In total this gives the following picture for $\mathfrak{h}(E^2)$.

$$
\begin{array}{ccccc}
 & & \wedge^2_K \mathfrak{h}^1(E) & & \\
 & \mathfrak{h}^1(E)^{\oplus 2} & & \mathfrak{h}^1(E)\big(-1\big)^{\oplus 2} & \\
\mathbf{1} & & \mathbf{1}(-1)^{\oplus 4} & & \mathbf{1}(-2)
\end{array}
$$

Each summand is a motive that contributes to $\mathrm{CH}(E^2)$ in only one degree, and we have placed the summands in a diagram of the type discussed in Section 8 of the Lecture Notes.

Moving on to E^3, the picture for $\mathfrak{h}(E^3)$ should be as follows.

$$
\begin{array}{ccccccc}
 & & & \wedge^3_K \mathfrak{h}^1(E) & & & \\
 & & \big[\wedge^2_K \mathfrak{h}^1(E)\big]^{\oplus 3} & & \big[\wedge^2_K \mathfrak{h}^1(E)\big(-1\big)\big]^{\oplus 3} & & \\
 & \mathfrak{h}^1(E)^{\oplus 3} & & \mathfrak{h}^1(E)\big(-1\big)^{\oplus 9} & & \mathfrak{h}^1(E)\big(-2\big)^{\oplus 3} & \\
\mathbf{1} & & \mathbf{1}(-1)^{\oplus 9} & & \mathbf{1}(-2)^{\oplus 9} & & \mathbf{1}(-3)
\end{array}
$$

In the mth horizontal layer what you see appearing, ignoring Tate twists, is the tensor product of a single motive, namely $\wedge^m_K \mathfrak{h}^1(E)$, with the $(3-m)$th exterior power of the standard representation of the Lie algebra $\mathfrak{gl}_6$. This is an instance of the refined Lefschetz decomposition of [11], Theorem 7.2, which I can explain to you in greater detail. What I guess is true but have not verified is that, for arbitrary g, the only 'new' motives that appear are the motives $\wedge^g_K \mathfrak{h}^1(E)$, and that $\mathrm{CH}\big(\wedge^g_K \mathfrak{h}^1(E)\big) = \mathrm{CH}^g\big(\wedge^g_K \mathfrak{h}^1(E)\big)$. (But I could be wrong here!) It would be very interesting to work this out in detail.

3. Modified diagonals and the curve class

3.1. We start by setting up some notation that will be in force throughout this project. We work over an algebraically closed field k of arbitrary characteristic. Let C/k be a smooth projective (irreducible) curve of genus g; we assume $g \geq 2$. Let J be the Jacobian of C.

We need some notation related to projection maps. If $n \geq 1$ and $I \subset \{1, \ldots, n\}$ is a subset with m elements, let $\mathrm{pr}_I \colon C^n \to C^m$ be the projection maps onto the factors indexed by I; thus, for instance, $\mathrm{pr}_i \colon C^n \to C$ is the projection to the ith factor, $\mathrm{pr}_{ij} \colon C^n \to C^2$ is the projection onto the ith and jth factors, etc. It will also be convenient to introduce the notation $\widehat{\mathrm{pr}}_I \colon C \to C^{n-m}$ for the projection to the factors that are not in I; thus, for instance, $\widehat{\mathrm{pr}}_i \colon C^n \to C^{n-1}$ is the projection map that omits the ith factor.

For $n \geq 1$, we denote by $\Delta_C^{(n)} \colon C \to C^n$ the diagonal morphism $x \mapsto (x, \ldots, x)$. For $n = 1$ this is the identity map on C; for $n = 2$ it is the usual diagonal map, and we write Δ_C instead of $\Delta_C^{(2)}$. We use the same notation $\Delta_C^{(n)}$ for the image of this diagonal morphism, which is called the small diagonal in C^n.

3.2. This project is about the relation between two invariants. We shall first explain this in the classical setting. As we shall see, this setup is too restrictive; a more general version will be discussed below. All Chow groups that we consider are with $\mathbb{Q}$-coefficients, so $\mathrm{CH}(X)$ from now on means $\mathrm{CH}(X) \otimes \mathbb{Q}$.

If we choose a base point $x_0 \in C(k)$, we get an embedding $\iota_{x_0} \colon C \hookrightarrow J$, given by $x \mapsto \mathscr{O}_C(x - x_0)$. Let $[C] \in \mathrm{CH}_1(J) = \mathrm{CH}^{g-1}(J)$ be the class of the curve $\iota_{x_0}(C) \subset J$, and let $[C]_{(s)}$ be the component of this class in the Beauville summand $\mathrm{CH}^{g-1}_{(s)}(J)$. It is known that

$$[C]_{(s)} = 0 \implies [C]_{(t)} = 0 \text{ for all } t \geq s. \tag{3.2.1}$$

Define $s(C, x_0)$ to be the smallest natural number s for which $[C]_{(s)} = 0$. It is known that $[C]_{(0)} \neq 0$, whereas $[C]_{(s)} = 0$ for all $s \geq g$, so $s(C, x_0) \in \{1, \ldots, g\}$.

There is a very famous result of Ceresa [5] that needs to be mentioned here, as it has had an enormous impact on research in this area. In the above setting, write $[C^-] \in \mathrm{CH}^{g-1}(J)$ for the image of $[C]$ under $[-1]_{J,*}$. Then it is easy to see that $[C]$ and $[C^-]$ have the same class in cohomology. Ceresa proved (working over $\mathbb{C}$) that for a very general curve C of genus ≥ 3, the classes $[C]$ and $[C^-]$ are *not* algebraically equivalent. It turns out to be rather subtle to decide for which curves C these classes are algebraically equivalent, and if we work modulo rational equivalence, the problem is of course even more subtle. (Note that modulo rational equivalence, the question depends on the chosen base point x_0; modulo algebraic equivalence the choice of x_0 does not matter.) Because $[C] - [C^-]$ is twice the sum of the terms $[C]_{(s)}$ for s running over the *odd* positive integers, and because we have the relation (3.2.1), we see that $[C] = [C^-]$ in $\mathrm{CH}^{g-1}(J)$ if and only if $[C]_{(1)} = 0$.

Let us now look at some other interesting classes, known as the modified diagonal classes, which were first studied in a paper by Gross and Schoen. We denote them by $\Gamma^n(C, x_0) \in \mathrm{CH}_1(C^n)$. We start with some explicit examples:

$$\Gamma^2(C, x_0) = [\Delta_C] - [C \times x_0] - [x_0 \times C] \quad \in \mathrm{CH}_1(C^2)\,,$$

and

$$\Gamma^3(C,x_0) = [\Delta_C^{(3)}] - \underbrace{[\Delta_C \times x_0]}_{+\text{ permutations}} + \underbrace{[C \times x_0 \times x_0]}_{+\text{ permutations}} \quad \in \mathrm{CH}_1(C^3)\,.$$

(In total $\Gamma^3(C,x_0)$ is a sum of $1+3+3$ terms.) The general formula is:

$$\Gamma^n(C,x_0) = \sum_{k=1}^{n} (-1)^{n-k} \cdot \Big(\sum_{\#I=k} \mathrm{pr}_I^*[\Delta_C^{(k)}] \cdot \prod_{j\notin I} \mathrm{pr}_j^*[x_0] \Big)\,, \tag{3.2.2}$$

where the second sum ranges over all subsets $I \subset \{1,\dots,n\}$ of cardinality k.

It is known that $\Gamma^n(C,x_0) = 0 \implies \Gamma^{n+1}(C,x_0) = 0$. Now define $n(C,x_0)$ to be the smallest integer n for which $\Gamma^n(C,x_0) = 0$. It turns out that there is a simple and beautiful connection with the invariant $s(C,x_0)$, which is given by the relation $n(C,x_0) = 2+s(C,x_0)$. Understanding the details of this already requires a considerable effort. (See for instance the proof of Theorem 4.3 in [13], which makes essential use of the results in [12].)

3.3. We can now finally explain what the project is about. In the above, we have used a base point $x_0 \in C(k)$ to embed the curve C into its Jacobian. It turns out that this is too restrictive. For instance, it is known that $\Gamma^3(C,x_0)$ and $[C]_{(1)}$ can vanish only if $(2g-2)\cdot x_0$ is a canonical divisor on C, which is an extremely restrictive condition. For this reason, we should consider a more general situation, in which we use an arbitrary 0-cycle $\xi \in \mathrm{CH}_0(C)$ of degree 1 to embed C in J. In that setting it is known that the vanishing of $\Gamma^3(C,\xi)$ (to be defined, see below) is again equivalent to the vanishing of the class $[C]_{(1)}$. However, for the relation between $\Gamma^n(C,\xi)$ and $[C]_{(n-2)}$ in general, not much seems known. As we shall discuss, there are in fact three different versions of modified diagonals, which we shall call $A^n(C,\xi)$, $B^n(C,\xi)$ and $\Gamma^n(C,\xi)$ such that:

- If we take $\xi = x_0$, a point, all three versions coincide with the classical Gross–Schoen modified diagonals $\Gamma^n(C,x_0)$.
- All three versions are the same modulo algebraic equivalence.
- For $n=3$ the vanishing of these classes is independent of which of the three we use.

Part of the project is to decide which of the three is the 'right' generalization of the Gross–Schoen modified diagonals.

3.4. Chow motives, the covariant theory. As a preparation for what follows, we briefly review the *covariant* theory of Chow motives, which turns out to be the most convenient setting to use. We denote this category of Chow motives by $\mathsf{CHM}_\bullet(k)$, where the lower dot should remind us that we use the 'homological' version.

If X and Y are smooth projective over k, define $\mathrm{Corr}_i(X,Y) = \mathrm{CH}_{\dim(X)+i}(X\times Y)$. Composition of correspondences gives us a map

$$\mathrm{Corr}_i(X,Y) \times \mathrm{Corr}_j(Y,Z) \to \mathrm{Corr}_{i+j}(X,Z)\,,$$

defined by $(\alpha,\beta) \mapsto \beta\circ\alpha = \mathrm{pr}_{XZ,*}\big(\mathrm{pr}_{XY}^*(\alpha)\cdot \mathrm{pr}_{YZ}^*(\beta)\big)$. The objects of $\mathsf{CHM}_\bullet(k)$ are triples (X,p,m) with X/k smooth projective, $p \in \mathrm{Corr}_0(X,X)$ a projector (meaning: $p\circ p = p$)

and $m \in \mathbb{Z}$. The morphisms are given by:

$$\mathrm{Hom}_{\mathsf{CHM}_\bullet(k)}\big((X,p,m),(Y,q,n)\big) = q \circ \mathrm{Corr}_{m-n}(X,Y) \circ p\,.$$

We have a covariant functor $\mathfrak{h}$ that sends X to $\mathfrak{h}_\bullet(X) = (X,[\Delta],0)$ and that sends $f\colon X \to Y$ to $[\Gamma_f] \in \mathrm{Corr}_0(X,Y) = \mathrm{Hom}_{\mathsf{CHM}_\bullet(k)}\big(\mathfrak{h}_\bullet(X),\mathfrak{h}_\bullet(Y)\big)$.

We define $\mathbf{1}(n) = \big(\mathrm{Spec}(k),\mathrm{id},n\big)$. If M is an object of $\mathsf{CHM}_\bullet(k)$, its Chow groups are defined by $\mathrm{CH}_n(M) = \mathrm{Hom}_{\mathsf{CHM}_\bullet(k)}\big(\mathbf{1}(n),M\big)$. Note that for a smooth projective X/k this gives

$$\mathrm{CH}_n\big(\mathfrak{h}_\bullet(X)\big) = \mathrm{Hom}_{\mathsf{CHM}_\bullet(k)}\big(\mathbf{1}(n),\mathfrak{h}_\bullet(X)\big) = \mathrm{Corr}_n\big(\mathrm{Spec}(k),X\big) = \mathrm{CH}_n(X)\,.$$

Here is an example that we shall use throughout the discussion that follows. Let C/k be a curve as above. If $\xi \in \mathrm{CH}_0(C)$ has degree 1, we obtain a motivic decomposition

$$\mathfrak{h}_\bullet(C) = \mathfrak{h}_0(C) \oplus \mathfrak{h}_1(C) \oplus \mathfrak{h}_2(C) \quad \text{with} \quad \mathfrak{h}_i(C) = (C,\pi_i,0)\,, \tag{3.4.1}$$

where the projectors $\pi_i \in \mathrm{Corr}_0(C,C) = \mathrm{CH}_1(C\times C)$ are given by

$$\pi_0 = C \times \xi\,, \qquad \pi_2 = \xi \times C\,, \qquad \pi_1 = [\Delta_C] - \pi_0 - \pi_2\,.$$

Moreover, $\mathfrak{h}_0(C) \cong \mathbf{1}$ and $\mathfrak{h}_2(C) \cong \mathbf{1}(1)$. This decomposition depends on the chosen ξ, but in what follows we shall not include this in the notation.

Another example that is relevant for us is that we have the (homological version of the) Deninger–Murre decomposition, i.e., we have a decomposition

$$\mathfrak{h}_\bullet(J) = \mathfrak{h}_0(J) \oplus \mathfrak{h}_1(J) \oplus \cdots \oplus \mathfrak{h}_{2g-1}(J) \oplus \mathfrak{h}_{2g}(J)$$

which is stable under all operators $[n]_*$, for $n \in \mathbb{Z}$, and such that $[n]_*$ induces $n^i \cdot \mathrm{id}$ on $\mathfrak{h}_i(J)$.

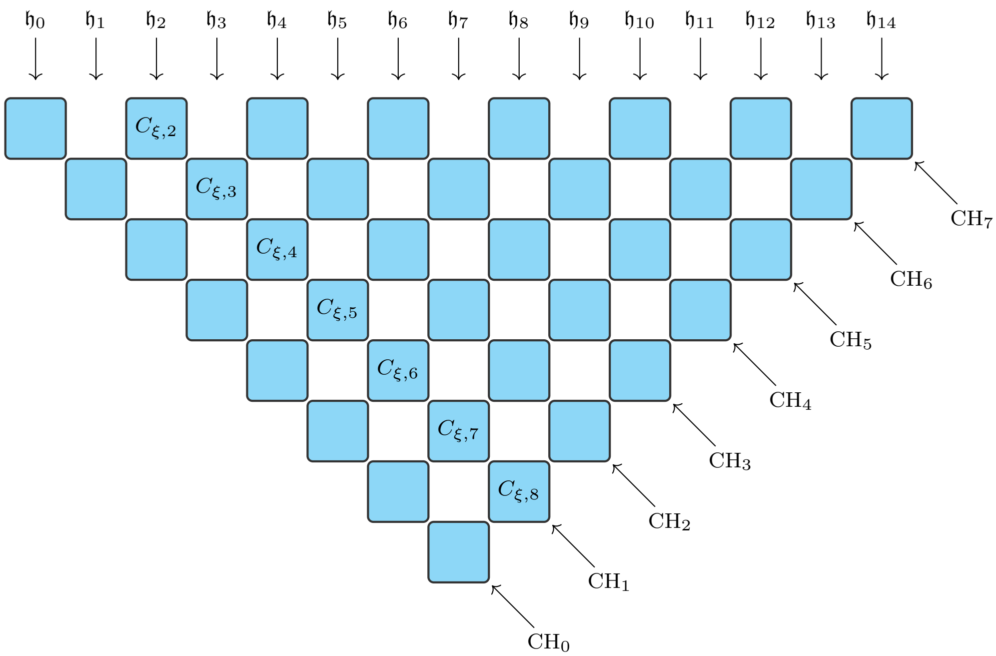
3.5. In everything that follows, ξ denotes an element of $\mathrm{CH}_0(C)$ of degree 1. Since ξ will usually be fixed, we will sometimes suppress it from the notation.

The class ξ induces a morphism of Chow motives $\iota_\xi\colon \mathfrak{h}_\bullet(C) \to \mathfrak{h}_\bullet(J)$, and hence homomorphisms $\iota_{\xi,*}\colon \mathrm{CH}_i(C) \to \mathrm{CH}_i(J)$. If ξ is represented by a 0-cycle on C with integral coefficients, say $\xi = \big[\sum_{j=1}^r\ m_j \cdot a_j\big]$ with $a_j \in C(k)$ and $m_j \in \mathbb{Z}$ (and of course $\sum m_j = 1$) then it is clear how to proceed: in this case, ξ defines an embedding $\iota_\xi\colon C \hookrightarrow J$ by $x \mapsto \mathscr{O}_C(x-\xi) \in J(k)$ and we simply consider the induced map on motives. A subtle point here is that this map on motives is well-defined for any $\xi \in \mathrm{CH}_0(C)_{\mathrm{deg}=1}$. (Recall that all Chow groups are taken with $\mathbb{Q}$-coefficients.) One way to see this is to note that every class in $\mathrm{CH}_0(C)_{\mathrm{deg}=0} = J(k)\otimes\mathbb{Q}$ can be represented by a 0-cycle on C with integral coefficients, because $J(k)$ is a divisible group (use that $k = \bar{k}$); this readily implies that also every class in $\mathrm{CH}_0(C)_{\mathrm{deg}=1}$ can be represented by a 0-cycle with integral coefficients. Moreover, if ξ and ξ' are 0-cycles of degree 1 with integral coefficients that define the same class in $\mathrm{CH}_0(C)$ then the resulting embeddings ι_ξ and $\iota_{\xi'}$ differ by a translation over a torsion point of J; but such a translation induces the identity on $\mathfrak{h}_\bullet(J)$. (For this last fact, cf. [2], Proposition 4(ii); it would be good to write out the details.) The morphism $\iota_\xi \colon \mathfrak{h}_\bullet(C) \to \mathfrak{h}_\bullet(J)$ that we obtain is therefore independent of which integral representative for the class ξ we choose.

For $i \geq 0$, define

$$C_{\xi,i} = \text{component of } \iota_{\xi,*}[C] \in \mathrm{CH}_1(J) \text{ in } \mathrm{CH}_1\big(\mathfrak{h}_i(J)\big).$$

It is easy to see that $C_{\xi,i}$ can be nonzero only for $2 \leq i \leq g+1$. If $\xi = x_0$ is a base point, the class $C_{\xi,i}$ is the same as the Beauville component $[C]_{(i-2)} \in \mathrm{CH}(J)$ considered in 3.2. The picture to keep in mind is as follows; this is the homological version of the picture discussed in Section 8 of the Lecture Notes. (In the picture we take $g = 7$ and we let $\mathfrak{h}_i = \mathfrak{h}_i(J)$.)



3.6. Problem 1. Is the analogue of (3.2.1) valid in the setting where we allow ξ to be an arbitrary element of $\mathrm{CH}_0(C)$ of degree 1 ? In other words, is it true that if $C_{\xi,i} = 0$ for some i then also $C_{\xi,j} = 0$ for all $j \geq i$?

The implication (3.2.1) was proven by Polishchuk in [15]. The argument uses the $\mathfrak{sl}_2$-action on $\mathrm{CH}(J)$ as in Theorem 9.4 from the Lecture Notes. Let $\lambda \in \mathrm{CH}^{g-1}_{(0)}(J)$ be the class as in that theorem; in the present setting, applying Theorem 9.4 to the Jacobian J of a curve C, we simply have $\lambda = [C]_{(0)}$. (Though the components $[C]_{(s)}$ for $s \geq 1$ in general depend on the chosen base point $x_0 \in C(k)$, the class $[C]_{(0)} \in \mathrm{CH}^{g-1}_{(0)}(J)$ is independent of choices.) In the present setting, the class ℓ of Theorem 9.4 of the Lecture Notes is the class $\theta \in \mathrm{CH}^1_{(0)}(J)$ of a symmetric theta divisor on J.

Polishchuk defines classes $p_n \in \mathrm{CH}^n_{(n-1)}(J)$ and $q_n \in \mathrm{CH}^n_{(n)}(J)$ by

$$p_n = \mathscr{F}\big([C]_{(n-1)}\big)\,, \qquad q_n = \mathscr{F}\big(\theta \cdot [C]_{(n)}\big)\,.$$

He then shows that the operator $f\colon y \mapsto \lambda \star y$ from the $\mathfrak{sl}_2$-action acts on polynomial expressions in these classes (using the ring structure of $\mathrm{CH}(J)$ given by the intersection product) as the operator

$$\mathscr{D} = \tfrac{1}{2} \cdot \sum_{m,n\geq 1} \binom{m+n}{n} \cdot p_{m+n-1} \cdot \partial_{p_m}\partial_{p_n} + \sum_{m,n\geq 1} \binom{m+n-1}{n} \cdot q_{m+n-1} \cdot \partial_{q_m}\partial_{p_n} - \sum_{n\geq 1} q_{n-1} \cdot \partial_{p_n}\,,$$

where ∂_{p_m} means 'take the partial derivative with respect to p_m (thought of as a variable)', and likewise for ∂_{q_m}. In particular,

$$f(p_2 p_m) = \binom{m+2}{2} \cdot p_{m+1} - q_1 p_m - q_{m-1} p_2\,, \tag{3.6.1}$$

and because it is clear from the definitions that $p_m = 0 \implies q_{m-1} = 0$, we obtain the implication $p_m = 0 \implies p_{m+1} = 0$, which is the Fourier-dual version of (3.2.1).

To deal with the general case (arbitrary ξ of degree 1), one option is to redo all of Polishchuk's calculations from [15]. This seems quite a bit of work. As what he proves goes much further than what we end up using, one could also try to give a direct proof of the relation (3.6.1), which is in fact Fourier-dual to the relation

$$\binom{m+2}{2} \cdot [C]_{(m)} = \big(\theta \cdot [C]_{(1)}\big) \star [C]_{(m-1)} + [C]_{(1)} \star \big(\theta \cdot [C]_{(m-1)}\big) - \theta \cdot \big([C]_{(1)} \star [C]_{(m-1)}\big)\,.$$

(At this point, it seems a good idea to remark that the RHS is what Polishchuk in [15] calls $[C]_{(1)} \star_1 [C]_{(m-1)}$ and to use ibid., Lemma 2.1.)

3.7. Modified diagonals—the first two variants. We keep the setting as before, so ξ is a class in $\mathrm{CH}_0(C)$ of degree 1. The first type of modified diagonal is an immediate generalization of (3.2.2). Define

$$\Gamma^n(C,\xi) = \sum_{k=1}^{n} (-1)^{n-k} \cdot \Big(\sum_{\#I=k} \mathrm{pr}_I^*[\Delta_C^{(k)}] \cdot \prod_{j\notin I} \mathrm{pr}_j^*\xi \Big)\,,$$

where the second sum ranges over all subsets $I \subset \{1,\ldots,n\}$ of cardinality k. For instance,

$$\Gamma^2(C,\xi) = [\Delta_C] - [C\times\xi] - [\xi\times C] \quad \in \mathrm{CH}_1(C^2)\,,$$

and

$$\Gamma^3(C,\xi) = \big[\Delta_C^{(3)}\big] - \underbrace{\big([\Delta_C]\times\xi\big)}_{+\text{ permutations}} + \underbrace{\big([C]\times\xi\times\xi\big)}_{+\text{ permutations}} \quad \in \mathrm{CH}_1(C^3)\,.$$

These modified diagonals have a nice motivic interpretation, which was first explored in [13]. We have the decomposition (3.4.1). Now define $\pi_+ = \pi_1 + \pi_2 = [\Delta_C] - (C\times\xi)$. This gives a coarser motivic decomposition $\mathfrak{h}(C) = \mathfrak{h}_0(C) \oplus \mathfrak{h}_+(C)$ with $\mathfrak{h}_+(C) = \mathfrak{h}_1(C) \oplus \mathfrak{h}_2(C)$. This induces decompositions

$$\mathfrak{h}(C^n) = \big(\mathfrak{h}_0(C) \oplus \mathfrak{h}_+(C)\big)^{\otimes n} = \bigoplus_{J \subset \{1,\ldots,n\}} \mathfrak{h}_J(C^n) \tag{3.7.1}$$

where, for $J \subset \{1, \ldots, n\}$ we define

$$\mathfrak{h}_J(C^n) = \mathfrak{h}_{\nu_1}(C) \otimes \mathfrak{h}_{\nu_2} \otimes \cdots \otimes \mathfrak{h}_{\nu_n}(C) \quad \text{with} \quad \nu_i = \begin{cases} + & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

In particular, the direct summand $\mathfrak{h}_{\{1,\ldots,n\}}(C^n) = \mathfrak{h}_+(C) \otimes \mathfrak{h}_+(C) \otimes \cdots \otimes \mathfrak{h}_+(C)$ is cut out by the projector $\pi_+^{\otimes n}$. With this notation,

$$\Gamma^n(C, \xi) = \big(\pi_+^{\otimes n}\big)_* \big[\Delta_C^{(n)}\big]\,,$$

i.e., the modified diagonal class $\Gamma^n(C, \xi)$ is nothing but the component of the class $\big[\Delta_C^{(n)}\big] \in \mathrm{CH}_1(C^n) = \oplus_J \mathrm{CH}_1\big(\mathfrak{h}_J(C^n)\big)$ in $\mathrm{CH}_1\big(\mathfrak{h}_+(C)^{\otimes n}\big)$.

This immediately brings us to the second type of modified diagonal class. Namely, instead of projecting the small diagonal class $\big[\Delta_C^{(n)}\big]$ to the summand $\mathfrak{h}_+(C)^{\otimes n}$, we could project to the even smaller summand $\mathfrak{h}_1(C)^{\otimes n}$. Indeed, we have the finer decomposition

$$\mathfrak{h}(C^n) = \big(\mathfrak{h}_0(C) \oplus \mathfrak{h}_1(C) \oplus \mathfrak{h}_2(C)\big)^{\otimes n} = \bigoplus_{\substack{K,L \subset \{1,\ldots,n\} \\ K \cap L = \emptyset}} \mathfrak{h}_{K,L}(C^n) \tag{3.7.2}$$

where we now define $\mathfrak{h}_{K,L}(C^n) = \mathfrak{h}_{\nu_1}(C) \otimes \mathfrak{h}_{\nu_2} \otimes \cdots \otimes \mathfrak{h}_{\nu_n}(C)$ with $\nu_i = 1$ if $i \in K$ and $\nu_i = 2$ if $i \in L$, and with $\nu_i = 0$ for $i \notin K \cup L$. The component $\mathfrak{h}_{\{1,\ldots,n\},\emptyset}(C^n) = \mathfrak{h}_1(C)^{\otimes n}$ is cut out by the projector $\pi_1^{\otimes n}$, and we are led to define

$$B^n(C, \xi) = \big(\pi_1^{\otimes n}\big)_* \big[\Delta_C^{(n)}\big]\,.$$

In other words, $B^n(C, \xi)$ is the component of $\big[\Delta_C^{(n)}\big]$ in $\mathrm{CH}_1\big(\mathfrak{h}_1(C)^{\otimes n}\big)$. First examples: $B^2(C, \xi) = \Gamma^2(C, \xi)$ and

$$B^3(C, \xi) = \Gamma^3(C, \xi) - \sum_{j=1}^{3} \widehat{\mathrm{pr}}_j^*\big(\Delta_{C,*}(\xi) - \xi \times \xi\big)\,. \tag{3.7.3}$$

A simple but important observation should be made here: for a point $x_0 \in C(k)$, viewed as a 0-cycle on C, we have $\Delta_{C,*}(x_0) = x_0 \times x_0$; but for a general 0-cycle, $\Delta_{C,*}(\xi)$ is not at all the same as $\xi \times \xi$. (Note that $\xi \mapsto \Delta_{C,*}(\xi)$ is a linear map, whereas $\xi \mapsto \xi \times \xi$ is quadratic.)

The relation (3.7.3) can be generalized. To express this, it is useful to define homomorphisms

$$\beta^n\,, \gamma^n \colon \mathrm{CH}(C) \to \mathrm{CH}(C^n)$$

by

$$\beta^n = \pi_1^{\otimes n} \circ \Delta_{C,*}^{(n)}\,, \qquad \gamma^n = \pi_+^{\otimes n} \circ \Delta_{C,*}^{(n)}\,.$$

(These maps of course depend on C and ξ but if there is no risk of confusion, we omit ξ from the notation.) As an example, if $\alpha \in \mathrm{CH}_0(C)$ then we have $\gamma^2(\alpha) = \beta^2(\alpha) = \Delta_{C,*}(\alpha) - \alpha \times \alpha$.

By definition, $B^n(C,\xi) = \beta^n[C]$ and $\Gamma^n(C,\xi) = \gamma^n[C]$. It is not hard to show that for all $\alpha \in \mathrm{CH}_0(C)$ and $n \geq 1$ we have $\beta^n(\alpha) = \gamma^n(\alpha)$. Further,

$$B^n(C,\xi) = \Gamma^n(C,\xi) - \sum_{j=1}^{n} \widehat{\mathrm{pr}}_j^*\bigl(\gamma^{n-1}(\xi)\bigr)\,. \tag{3.7.4}$$

(To get familiar with these notions, you should of course prove these assertions!) It is also not hard to show (see [13], Remark 2.6) that if we take a point $x_0 \in C(k)$ we have $\gamma^{n-1}(x_0) = 0$, so that $B^n(C,x_0) = \Gamma^n(C,x_0)$ is the 'usual' Gross–Schoen modified diagonal class.

3.8. Problem 2. First show that

$$\Gamma^n(C,\xi) = 0 \iff B^n(C,\xi) = 0 \text{ and } \gamma^{n-1}(\xi) = 0\,.$$

Note that (3.7.2) is a refinement of the decomposition (3.7.1). It helps if you understand in which summands of (3.7.2) all terms that occur in (3.7.4) are located.

For $n = 3$, something extremely interesting happens. Namely, the following can be proven.

- If $B^3(C,\xi) = 0$ then $(2g-2)\cdot\xi = K_C$.
- If $(2g-2)\cdot\xi = K_C$ then $B^3(C,\xi) = 0$ implies that $\gamma^2(\xi) = 0$, and hence that $\Gamma^3(C,\xi) = 0$.

We conclude from these that the vanishing of $\Gamma^3(C,\xi)$ is in fact equivalent to the vanishing of $B^3(C,\xi)$, and that there is a unique element $\xi \in \mathrm{CH}_0(C)_{\deg=1}$ for which this may happen, namely $\xi = \frac{1}{2g-2}\cdot K_C$.

Try to prove all these assertions. Some hints: for the first assertion, consider $B^3(C,\xi)$ as a correspondence (of degree -1) from C^2 to C, and apply this correspondence to the class $[\Delta_C] \in \mathrm{CH}_1(C^2)$. For the second assertion, assume $(2g-2)\cdot\xi = K_C$, calculate the intersection product $B^3(C,\xi)\cdot([\Delta_C]\times C)$, and then apply $\mathrm{pr}_{23,*}$. (These calculations are not extremely hard but you need to do them carefully, as it is easy to make mistakes.)

3.9. In view of these results, there is a canonical choice for the 0-cycle ξ, namely $\xi = \frac{1}{2g-2}\cdot K_C$. With that choice, it is known that

$$\Gamma^3(C,\xi) = 0 \iff C_{\xi,3} = 0 \iff \iota_{\xi,*}[C] \in \mathrm{CH}_1\bigl(\mathfrak{h}_2(J)\bigr)\,. \tag{3.9.1}$$

(For the notation $C_{\xi,i}$ see 3.5.) This is due to Zhang [17], Theorem 1.5.5.[1]

[1] For some of the formulas presented in [17] the details are omitted, and my own calculations in fact give slightly different formulas. The same result (3.9.1) is also claimed, with a very different proof, in [7], Proposition 3.1, but the proof that is given there is not correct as it makes essential use of results from [9] that only work in a more restrictive setting and do not carry over to the situation where ξ is an arbitrary class in $\mathrm{CH}_0(X)_{\deg=1}$.

In general, it is very hard to decide for which curves C, taking $\xi = \frac{1}{2g-2} \cdot K_C$, the modified diagonal class $\Gamma^3(C,\xi)$ vanishes. Recently, several very nice papers have appeared that contain results about this; see for instance [4], [10], [16].

3.10. Problem 3. To what extent do these results for $n = 3$ have analogues for higher values of n ? By the above, the vanishing of $\Gamma^n(C,\xi)$ implies that also $\gamma^{n-1}(\xi) = 0$; what restrictions on ξ does this give? For $n \geq 4$, is it true that the vanishing of $B^n(C,\xi)$ is equivalent to the vanishing of $\Gamma^n(C,\xi)$, similar to what happens for $n = 3$, or does this at least hold if we take $\xi = \frac{1}{2g-2} \cdot K_C$?

3.11. Modified diagonals—a third variant. There is an obvious third way to generalize the Gross–Schoen modified diagonal classes, namely by defining

$$A^n(C,\xi) = \sum_{k=1}^{n} (-1)^{n-k} \cdot \Big(\sum_{\#I=k} \mathrm{pr}_I^*[\Delta_C^{(k)}] \cdot \widehat{\mathrm{pr}}_I^*\big(\Delta_{C,*}^{(n-k)}(\xi)\big) \Big)\,,$$

where again the inner sum runs over the subsets $I \subset \{1,\ldots,n\}$ of cardinality k, and where we recall that $\widehat{\mathrm{pr}}_I \colon C^n \to C^{n-k}$ denotes the projection onto the factors indexed by the complement of I. The difference with the classes $\Gamma^n(C,\xi)$ is that we are replacing all occurences of terms $\xi \times \cdots \times \xi$ (m factors, say) by $\Delta_{C,*}^{(m)}(\xi)$. Clearly, if ξ is a point, this makes no difference, so in that case $A^n(C,\xi)$ is the same as $\Gamma^n(C,\xi)$. More generally, we have $A^n(C,\xi) = \Gamma^n(C,\xi)$ whenever $\Delta_{C,*}(\xi) = \xi \times \xi$. We have $A^2(C,\xi) = B^2(C,\xi) = \Gamma^2(C,\xi) = [\Delta_C] - [C \times \xi] - [\xi \times C]$, and

$$A^3(C,\xi) = \big[\Delta_C^{(3)}\big] - \underbrace{\big([\Delta_C] \times \xi\big)}_{+\text{ permutations}} + \underbrace{\big([C] \times \Delta_{C,*}(\xi)\big)}_{+\text{ permutations}} \quad \in \mathrm{CH}_1(C^3)\,.$$

A nice feature of this variant is that $A^n(C,\xi)$ is 'linear in the class ξ', by which we mean that if $\xi = \sum_{i=1}^r m_i \cdot x_i$ then

$$A^n(C,\xi) = \sum_{i=1}^{r} m_i \cdot A^n(C,x_i)$$

as classes in $\mathrm{CH}_1(C^n)$.

3.12. Problem 4. Show that $A^3(C,\xi) = 0$ if and only if $\Gamma^3(C,\xi) = 0$. How about higher values of n ? Does the vanishing of $A^n(C,\xi)$ imply the vanishing of either $B^n(C,\xi)$ or $\Gamma^n(C,\xi)$, or the other way around? Does the vanishing of $A^n(C,\xi)$ imply the vanishing of $A^{n+1}(C,\xi)$?

3.13. Finally, we come to a question for which I don't even know what to expect. As we have seen, if we use a base point $x_0 \in C(k)$ to embed C into its Jacobian then the vanishing of $\Gamma^n(C,x_0)$ for some $n \geq 3$ is equivalent to the vanishing of the component $[C]_{(n-2)}$ of the curve class. (And, once again, in this case $A^n = B^n = \Gamma^n$.) On the other hand, we have seen that $\Gamma^3(C,x_0)$ can only vanish if $(2g-2) \cdot x_0$ is the canonical class, and while this may happen (hyperelliptic curves, plane quartic curves with a hyperflex, ...), this condition is very restrictive. It is for this reason that we want to consider a more general setup, allowing a class $\xi \in \mathrm{CH}_0(C)_{\deg=1}$ instead of a base point. Many results in the literature (or at least their proofs) are no longer valid in that setting. Further, we now have (at least) three candidates for the modified diagonal classes, and it is not at all clear which of the three has the best properties. Does one of the three versions have the property that the vanishing of the nth modified diagonal class is equivalent to the vanishing of the class $C_{\xi,n} \in \mathrm{CH}_1\big(\mathfrak{h}_n(J)\big)$ as in Section 3.5? It would be great to understand this better.

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