$$\mathfrak{sl}_{2,\mathbb{Q}} := \mathbb{Q}\cdot e \oplus \mathbb{Q}\cdot h \oplus \mathbb{Q}\cdot f$$

$$\left.\begin{array}{ll} [e,h] = 2e & [e,f] = h \\ [f,h] = -2f & \end{array}\right\} (*)$$

V: $\mathbb{Q}$-v.space, $\rho : \mathfrak{sl}_2 \longrightarrow \mathrm{End}(V)$:

$e, f, h \in \mathrm{End}(V)$ + $(*)$

St = standard rep $= \mathbb{Q}^2$

$$e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\mathrm{Sym}^n(St)$ $\quad$ $n+1$ dim'l repr.
irred.

this gives all irreps, $\dim < \infty$

$\mathrm{Sym}^n(\mathrm{St})$:

base vectors

$$w_{-n},\ w_{-n+2},\ \dots \qquad w_{n-2},\ w_n$$

$$h(w_j) = j \cdot w_j$$

$$e(w_{-n+2i}) = (n-i) \cdot w_{-n+2i+2}$$

$$f(w_{-n+2i}) = i \cdot w_{-n+2i-2}$$

Exa $n=5$

$$
\begin{tikzcd}
-5 \arrow[r, bend left, "e"] & -3 \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & -1 \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & 1 \arrow[l, bend left] \arrow[r, bend left, "e"] & 3 \arrow[l, bend left] \arrow[r, bend left, "e"] & 5 \arrow[l, bend left]
\end{tikzcd}
$$

primitive vector

X/k AV, $\dim = g$

$\theta : X \longrightarrow X^t$ polarization

$\theta = \varphi_L$ $\qquad L$ symmetric ample

$\ell := c_1(L) \in CH^1_{(0)}(X)$

$\theta \rightsquigarrow \quad \theta_* : CH(X)_{\mathbb{Q}} \xrightarrow{\sim} CH(X^t)_{\mathbb{Q}}$

$\lambda \in CH^{g-1}_{(0)}(X)$ unique class

s.t. $\theta_*(\lambda) = \mathcal{F}(\ell)$.

<u>Theorem</u>: Op. on $CH(X)_{\mathbb{Q}}$:

$e(\alpha) = \ell \cdot \alpha$ inters. prod

$h(\alpha) = (2i-s-g) \cdot \alpha$ if $\alpha \in CH^i_{(s)}$

$f(\alpha) = \lambda \star \alpha$ Pontr. prod

Then e, h, f satisfy the comm. rels $(*)$ & therefore define a repr. of $\mathfrak{sl}_2$ on $CH(X)_{\mathbb{Q}}$.

For every s, the subspace

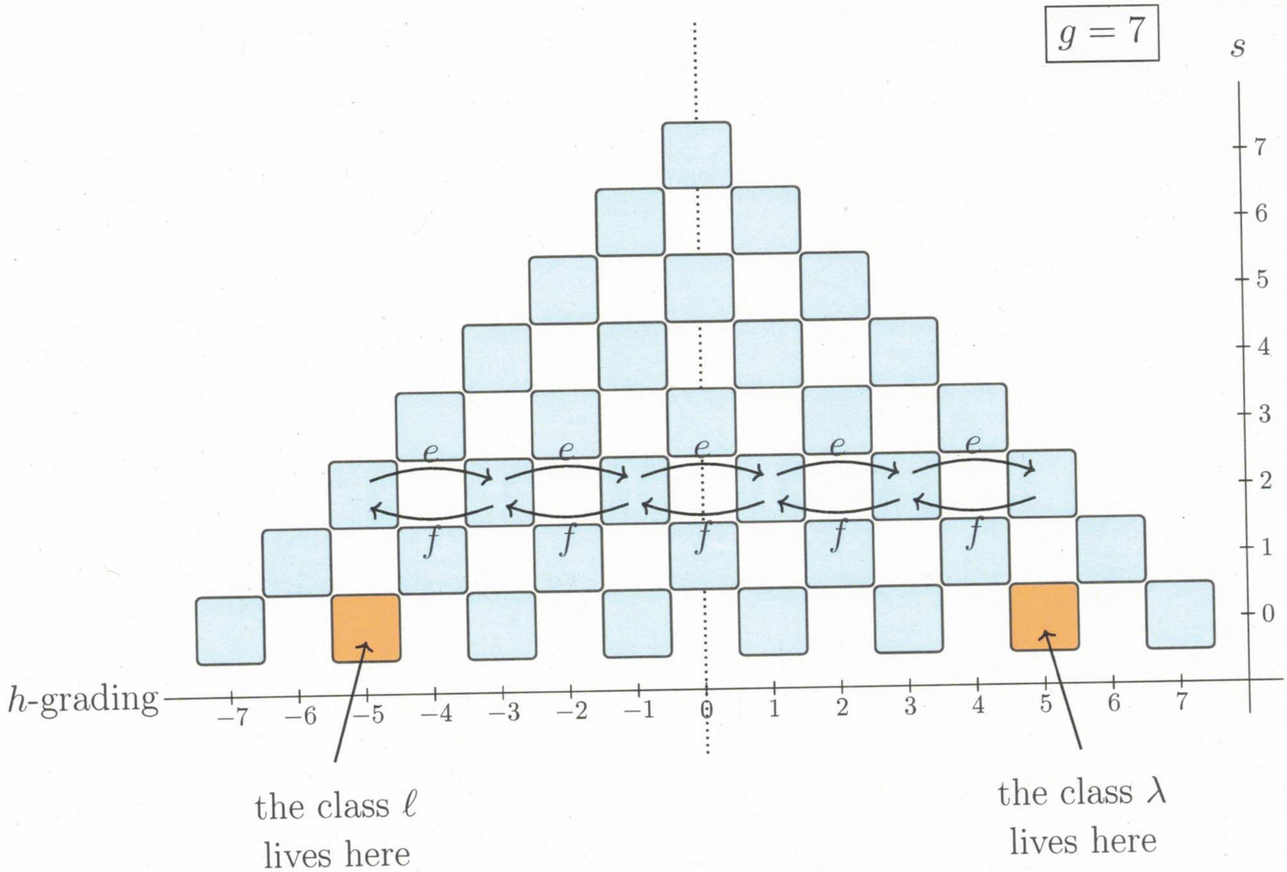
$$CH^{\bullet}_{(s)} := \bigoplus CH^{i}_{(s)}$$

is stable under $\mathfrak{sl}_2$ and:

$\exists$ $\mathbb{Q}$-subspaces $P_{j,s} \subseteq CH^{\cdots}_{(s)}$:

$$CH^{\bullet}_{(s)}(X) = \bigoplus_{\substack{j=0,\dots,g-|s| \\ j \equiv g-s\ (2)}} P_{j,s} \otimes \mathrm{Sym}^{j}(St)$$

$$e(x) = \ell \cdot x \qquad \text{(intersection product with the class } \ell\text{)}$$
$$h(x) = (2i - s - g) \cdot x \qquad \text{if } x \in \mathrm{CH}^i_{(s)}(X)$$
$$f(x) = \lambda \star x \qquad \text{(Pontryagin product with the class } \ell\text{)}$$



0-cycles

/ $k = \bar{k}$, work integrally

$CH_0(X)$: ring for $\star$-prod.

$$\deg : CH_0(X) \longrightarrow \mathbb{Z}$$

$$\sum m_i (P_i) \longmapsto \sum m_i$$

$$m \cdot (0) \longleftarrow m$$

$I :=$ ~~deg~~ $\mathrm{Ker}(\deg)$ $\star$-ideal

↪ as $\mathbb{Z}$-mod gen'd by all $(P) - (0)$.

$CH_0 \supset I \supset I^{\star 2} \supset I^{\star 3} \supset \dots$

Summation map $S : I \longrightarrow X(k)$:

$$\sum m_i (P_i) \longmapsto \underbrace{\sum m_i P_i}_{\text{use group law in } X} .$$

<u>Prop.</u> We have s.e.s.

$$0 \longrightarrow I^{\star 2} \longrightarrow I \xrightarrow{S} X(k) \longrightarrow 0$$

$I^{\star 2}$ gen'd by all $((P)-(0)) \star ((Q)-(0))$

$= (P+Q) - (P) - (Q) + (0)$

clear: $I^{\star 2} \subseteq \operatorname{Ker}(S)$.

$$((P)-(0)) + ((Q)-(0))$$

$$\equiv ((P+Q)-(0)) \mod I^{*2}$$

$\rightsquigarrow$ every class in I/I^{*2} is repr.'d by some $((P)-(0))$.

$\Rightarrow I^{*2} = \mathrm{Ker}(S)$. $\boxtimes$

Rk: $I = \{\alpha \in CH_0 \mid \alpha \underset{alg}{\sim} 0\}$

$\Rightarrow I$ is divisible

$\Rightarrow I^*$ is divisible

<u>Thm</u> (Roitman)

summ. map S induces

$$I_{tors} \xrightarrow{\sim} X(k)_{tors}$$

<u>Cor</u>: I^{*2} torsion free

+ divisible

hence it is a $\mathbb{Q}$-vector space!

$$CH_0(X) \supset I^{2} \supset I^{\star 2} \supset \cdots$$

$$CH_0(X)_{\mathbb{Q}} \supset I_{\mathbb{Q}} \supset I^{\star 2} \supset \cdots \qquad (A)$$

also:

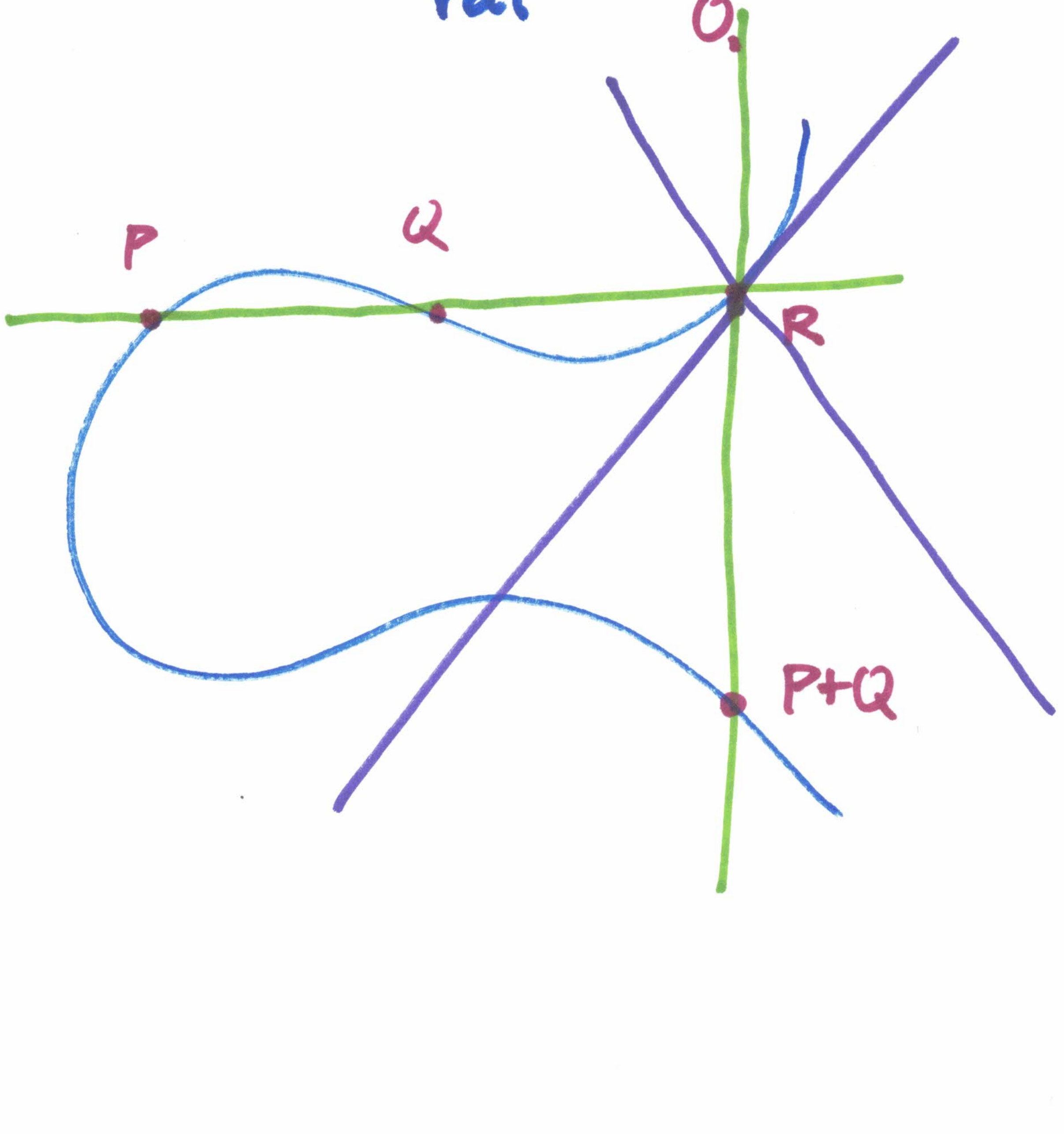
$$CH_0(X)_{\mathbb{Q}} \supset \bigoplus_{s \geq 1} CH_{0,(s)} \supset \bigoplus_{s \geq 2} CH_{0,(s)} \supset \cdots \qquad (B)$$

Theorem (A) and (B) are the same

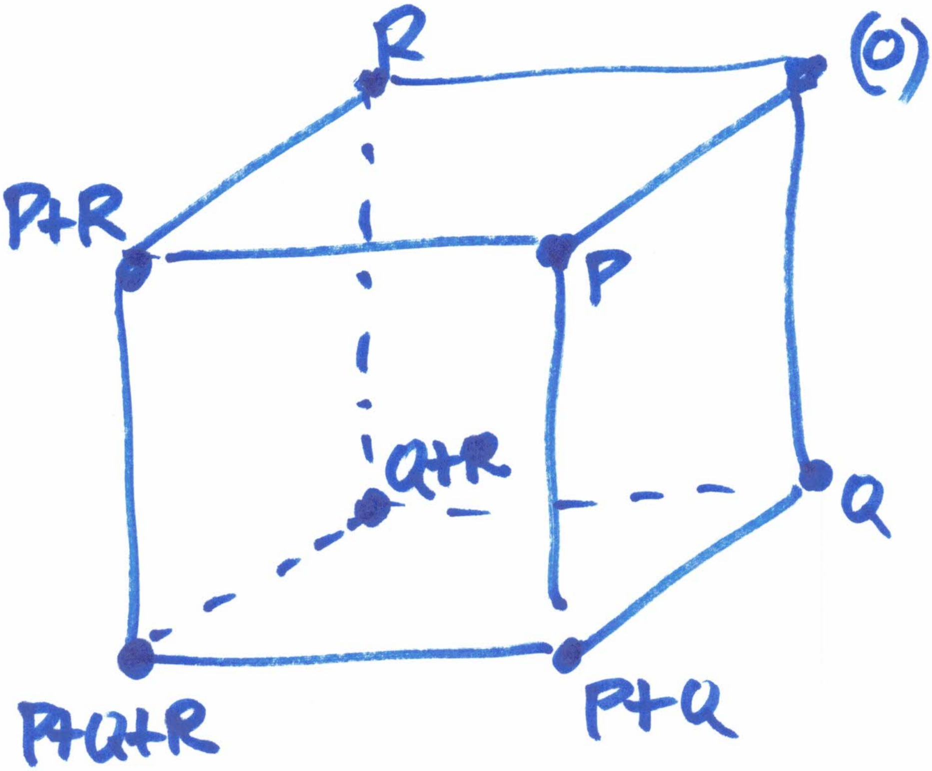
Cor: $I^{\star (g+1)} = 0$

Exa $g=1$: $\forall P, Q \in X$:

$$(P+Q) + (O) + (R) \underset{\text{rat}}{\sim} (P) + (Q) + (R)$$



Exa $g=2$ $\qquad$ $P, Q, R \in X$



$$(P+Q+R) + (P) + (Q) + (R)$$

$$\sim_{\mathrm{rat}} (P+Q) + (P+R) + (Q+R) + (0)$$

General g : $\quad I^{*(g+1)}$:

"hypercube" statement

<u>brief sketch of pf</u> :

- $I = \bigoplus_{s \geq 1} CH_{0,(s)}$

- Beauv. dec. compatible w/ $\star$-prod

  $\Rightarrow \quad I^{\star r} \subseteq \bigoplus_{s \geq r} CH_{0,(s)}$

  (enough for Corollary)

- $[n]_* I \subseteq I \quad \Rightarrow$

  $[n]_* I^{\star r} \subseteq I^{\star r} \quad \forall r$

- $[n]_*$ induces mult. by n on $I/I^{*2} \cong X(k)$

Since

$$(I/I^{*2})^{\otimes r} \twoheadrightarrow I^{*r}/I^{*r+1}$$

get: $[n]_* \curvearrowright I^{*r}/I^{*r+1}$ as mult by n^r

- "weight arg" $\Rightarrow$ (A) = (B)

$\boxtimes$