Algebraic Cycles on $AV$

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$/k$, $X, Y$ : sm. proj. var. $/k$

**Def:** $i, j \in \mathbb{Z}$

$Z^i(X) := \mathbb{Z} \cdot \left\{ \text{cl. irreducible subspace } \mathcal{Z}_C X \right\}$ of codim $i$

$Z_j(X) := \mathbb{Z} \cdot \left\{ \mathcal{Z} \mid \text{dim}(\mathcal{Z}) = j \right\}$

elt: $\sum \sum_{i \in \mathbb{Z}} Z_i^{m_i}$ $m_i \in \mathbb{Z}$

**Rat'l eq:** $W \subset X$ irreducible of Codim $= i - 1$

generated by: $0 \neq f \in k(W) \sim \text{div}(f)$

$\text{div}(f) \sim \text{rat } 0$
Flat fam. $U$ of cycles on $X \times \mathbb{P}^1$

at $t$: $V_t$

Rat'eq: gen'd by

$V_{t_1} \sim \text{rat} \; V_{t_2}$

$CH^i(X) := Z^i(X)/\sim \text{rat}$

$CH_j(X) := Z_j(X)/\sim \text{rat}$

$CH(X) := \bigoplus_i CH^i(X) = \bigoplus_j CH_j(X)$

$CH(X)_{\mathbb{Q}} := CH(X) \otimes \mathbb{Q}$
Exa (X ined)

\[ \text{ch}^0(X) \cong \mathbb{Z}. [x] \]

\[ \text{ch}^1(X) = \text{Cl}(X) \cong \text{Pic}(X) \]

Operations

**push-forward** \( f : X \rightarrow Y \) 

\[ f_* : \text{ch}^i(X) \rightarrow \text{ch}^i(Y) \]

**Idea**: \( ZcX \rightarrow f(Z)cY \)

- \( Z \rightarrow f(Z) : \) if gen. f.m. of deg = d
- then \( f_* [Z] = d. [f(Z)] \)
- else \( f_* [Z] = 0 \).
Pullback (Gysin) \( f: X \to Y \)
\[ f^*: CH^*(Y) \to CH^*(X) \]
preserves codim-grading

Special case: if flat then for \( W \subseteq Y \)
\[ f^{-1}(W) \to f^* [W] = [f^{-1}(W)]. \]

Intersection product: \( (X/\& \text{sm. proj}) \)

\( CH^*(X) \) is comm. graded ring

\[ CH^i(X) \times CH^j(X) \to CH^{i+j}(X) \]

Very special case: \( W, Z \subseteq X \) intersect transversally:
\[ [W] \cdot [Z] = [Z \cap W]. \]
Exterior prod: \( X \times Y / \mathcal{E} \)

\[ \alpha \in \text{CH}^i(X), \quad \beta \in \text{CH}^j(Y) \]

\[ \Rightarrow \alpha \times \beta \in \text{CH}^{i+j}(X \times Y) \]

Idea: \( \alpha = [\mathcal{W}] \) then \( \alpha \times \beta = [\mathcal{W} \times \mathcal{Z}] \)

\( \beta = [\mathcal{Z}] \)

Relations:

1. \( \alpha \cdot \beta = \Delta^* (\alpha \times \beta) \)

\[ \Delta: X \to X \times X \]

2. \( \alpha \times \beta = \pi_1^* (\alpha) \cdot \pi_2^* (\beta) \)

\[ \begin{array}{ccc}
\pi_1 & \downarrow & \pi_2 \\
X \times X & \to & X \\
\end{array} \]
Projected formula: \( f: X \to Y \)

\[
\psi_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)
\]

\[X/\mathcal{P} \text{ ab. var, dim }= q,\]
\[m: X \times X \to X\]

\[\square \text{ CH}(X) \text{ has a 2nd ring structure!}\]

\[\star : \text{CH}_i(X) \times \text{CH}_j(X) \to \text{CH}_{i+j}(X)\]
\[\alpha, \beta \mapsto m_*(\alpha \times \beta)\]

\[\text{CH}(X): \text{ comm. graded ring}\]

\[\uparrow \text{ for dim of cycles}\]
\[
\begin{align*}
X & \xrightarrow{\Delta} X \times X \xrightarrow{\mu} X \\
\alpha \times \beta \\
\alpha = [w], \quad \beta = [z] & \xrightarrow{\mu} \\
(w + z) \subset X \\
\{p + q \mid p \in \mathbb{G} w, \quad q \in \mathbb{Z}\} & \\
w \times z & \xrightarrow{\mu} (w + z) \quad \text{if gen. fin. of degree } = d \text{ then } \alpha \star \beta = d \cdot [(w + z)] \\
\text{unit for } \times \text{-prod} & = [e]
\end{align*}
\]
$X \rightarrow X^t := \text{Pic}^0_{X/k}$

$\text{Pic}^0_{X/k} = \text{moduli of line bun on } X$

$\text{Pic}^0 \text{ comp. } = \mathbb{G}_m$

Always: $X \sim X^t$

in gen'l: $X \not\sim X^t$

Poincane LB: $P$ on $X \times X^t$

$\exists \in X^t: P|_{X \times \{ \exists \}} = \text{line bun on } X$

corr. to $\exists$

$X \cong (X^t)^t$

$P_{X^t}$ on $X^t \times X$ is just $(\text{SW})^* P$

$\text{SW}: X^t \times X^t \rightarrow X \times X^t$
\( \eta := \zeta_1(P) \in CH^1(X \times X^t) \)

\[
ch(P) := \exp(\eta) = 1 + \eta + \frac{1}{2} \eta^2 + \cdots + \frac{1}{(2g)!} \eta^{2g} \in CH(X \times X^t)_{Q}
\]

**Def.** Fourier transform

\( \mathcal{F} = \mathcal{F}_X : CH(X)_{Q} \rightarrow CH(X^t)_{Q} \)
\[ F(x) = \Pr_{X^t, \star} \left( \Pr_X^*(x) \cdot \text{ch}(P) \right) \]

\[ F^t = F^t_{x^t} : \text{CH}(X^t)_Q \rightarrow \text{CH}(X)_Q \]

**THEOREM**: (Mukai, Beauville)

(i) \[ F^t \circ F = (-1)^t \cdot [-1]_* \]

\( (n \in \mathbb{Z}, [n]_X: X \rightarrow X \text{ mult by } n \text{ map} ) \)

Hence \( F: (\text{CH}(X)_{Q^*}) \sim (\text{CH}(X^t)_{Q^*}) \)

(ii) \[ F(\alpha \ast \beta) = F(\alpha) \cdot F(\beta) \]

\[ F(\alpha \cdot \beta) = (-1)^t \cdot F(\alpha) + F(\beta) \]