3. Geometry of $\mathcal{S}_g$

\[ \mathcal{S}_g = \overline{\mathbb{F}_p} \supset \mathbb{F}_q \supset \mathbb{F}_p, \mathbb{K} \]

Recall $E/K$ is supersingular if $E[p](K) = \{0\}$

$X/K$ is supersingular if $X \cong \mathbb{A}^g$, $E$ ss EC

$X/K$ is superspecial if $X \cong \mathbb{A}^g$, $E$ ss EC

**Thm 1.23 (Deligne)**

Any two products $\left(\prod_{i=1}^n\right)$

$E_1 \times \cdots \times E_n$, $E_{n+1} \times \cdots \times E_{2n}$

of ss EC's are isomorphic over $\mathbb{K}$ as abelian varieties.
From now on, we study

\[ S_g = \{ (X, \lambda) \in A_g : X \text{ is supersingular} \} \]

- This is a coarse moduli space of \( g \)-dim supersingular ppAV's.

- \( S_g = W_0 \subseteq A_g \) is a Newton stratum.

- When \( g = 1 \), \( S_g \) has dim 0
  (55 points on \( j \)-line)
Notation 3.2 Let $E_0$ be a ssEC defined over $\mathbb{F}_{p^2}$, with $\Pi E_0 = -\mathbf{p}$.

Fix $g$-dimensional superspecial (ssp) AV to be $E_0^g/\mathcal{A}$.

Idea Any (ss)pp AV may be related to a polarised (ssp)AV through some isogeny.

Every irr component of $\mathcal{B}_g$ is determined by a choice of polarised ssp AV.
\( q = 2 \) Any irr. component of \( S_2 \) is the image of a "Moret-Bailly family" \( q: (X, D) \rightarrow \mathbb{P}^1 \)

\[ 1 \rightarrow \alpha_p \times \mathbb{P}_k^1 \rightarrow A \times \mathbb{P}_k^1 \rightarrow \mathcal{E}^D \rightarrow 1 \]

\[ \pi^* M = \pi^* \mathcal{L} \]

\[ \mathcal{L}, \mathcal{A} \]

where \( A \) is a sssp surface, \( \mathcal{L} \) line bundle

\( \Leftrightarrow \lambda \) polarisation kernel \( \alpha_p \times \alpha_p \)

\[ q: (X, D) \rightarrow \mathbb{P}^1_k \]

So:

\( \cdot \) every irr. component of \( S_2 \) is a rational curve

\[ \Rightarrow \text{dim } S_2 = 1 \]

\( \cdot \) \# cpts = \# pols \( \lambda \) with kernel \( \alpha_p \times \alpha_p \)

\[ \Rightarrow \text{count tomorrow} \]
Let \( F = F_{x_1 \theta}, V = V_{x_1 \theta} \). Write \( X[F] = \ker(F) \) on \( X \).

For general \( g \), over \( \theta \):

**Def 3.6** A polarised flag type quotient (PFTQ) w.r.t. polarisation \( \mu \) on \( E_{\theta} \)

\[
\text{s.t. } \ker(\mu) = \begin{cases} E_{\theta} & g \text{ even} \\ 0 & g \text{ odd} \end{cases}
\]

is a chain of isogenies

\[
(\lambda_{g-1}, \beta_{g-1}) \xrightarrow{\phi_{g-1}} (\lambda_{g-2}, \beta_{g-2}) \rightarrow \ldots \rightarrow (\lambda_1, \beta_1) \xrightarrow{\phi_1} (\lambda_0, \beta_0)
\]

\[
\text{s.t.} \\
(\lambda_{g-1}, \beta_{g-1}) = (E_{\theta}, \rho_{\frac{g-1}{2}} \mu) \\
\ker(\phi_i) \cong \alpha_i \sigma_i^2 \quad \forall 1 \leq i \leq g-1 \\
\ker(\beta_i) \subseteq \mathbb{H}^0 \text{ EV}^{d_0} \text{ F}_{i-\delta}^{\leq i} \\
\quad \forall 0 \leq i \leq g-1 \\
\text{and } \beta_j \leq \frac{i}{2} \\
(\lambda_0, \beta_0) \text{ is a } \mathbb{D} \text{ ss pp AV}
\]
Back to $g=2$

PFTQ is

\[(E_0^2, \mu) \rightarrow (E_0^2/\alpha_\rho, \lambda_0) = (Y_0, \lambda_0)\]

\[\ker(\mu) = E_0^2 \mathbb{C}[F]\]

\[= \alpha_\rho \times \alpha_\rho\]

This is determined by

\[\alpha_\rho \leftrightarrow \alpha_\rho \times \alpha_\rho \leftrightarrow E_0 \times E_0\]

\[\text{End } (\alpha_\rho) = \mathbb{R}, \text{ so may view this as}\]

\[(a : b) \in \mathbb{P}^1_{\mathbb{R}}\]

So again \(\mathbb{P}^1\)-family!
Def 3.11 $g$-dim PFTO's w.r.t. $\mu$

have a (fine!) moduli space $\mathcal{P}_{g, \mu}$
defined over $\mathbb{F}_p^2$.

It is geometrically irreducible, quasi-projective
of dim $\lfloor \frac{g^2}{2} \rfloor$. 
Projection to last member gives
\[ \bigoplus_{\mu} P_{g,\mu} \to \mathcal{F}_g \]

Surjective & generically finite
(a number 1 \( \Rightarrow \exists! \) PFTQ above it)

So \( \text{dim } (\mathcal{F}_g) = \left\lceil \frac{A^2}{4} \right\rceil \).

- can also see this from \( \text{dim } (W_6) = |\Delta(6)| \) -

and \# in cpts of \( \mathcal{F}_g \) =

\# suitable polarisations \( \mu \)

\( \Rightarrow \) count tomorrow
Ex 3.13 (g=3)

$P_{3, \mu}$ has dim $\lfloor \frac{g}{4} \rfloor = 2$, structure indep. of $\mu$

$(y_2, \lambda_2) \rightarrow (y_1, \lambda_1) \rightarrow (y_0, \lambda_0) = y$

\[ a = 1 \]

$\geq 2$

$\geq 3$

\[ \pi \]

$P^1$-bundle

$CC \mathbb{P}^2 \xrightarrow{\text{Wirtinger}} (y_2, \lambda_2) \rightarrow (y_1, \lambda_1)$

$C: x_1^{p+1} + x_2^{p+1} + x_3^{p+1} = 0$
Away from $T$, $y \mapsto (y_0, x_0)$ is generically finite.

- $y$ is determined by 2 parameters:
  
  \[
  t = \pi(y) \in C(\mathbb{R}) \\
  u \in \pi^{-1}(t) = \mathbb{P}^1_{\mathbb{R}}(\mathbb{R})
  \]

- $y \in T \Rightarrow a(y) := a(y_0) = 3$
  
  \[
  t \in C(\mathbb{F}_p^2) \Rightarrow a(y) \geq 2 \\
  a(y) = 3 \iff u \in \mathbb{P}^1_{\mathbb{F}_p^2}(\mathbb{F}_p^2)
  \]

- $a(y) = 1 \iff y \notin T, t \in C(\mathbb{F}_p^2)$
Thm 3.26 For any irreducible $V$ of $fg$, a finite surjective morphism $\Phi : D \times J \to V$ s.t. any $\Phi(D \times fj)$ is a central leaf and any $\Phi(fdx \times J)$ is an isogeny leaf (all leaves are found this way)

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central leaves
isogeny leaves
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"almost-product structure"
Def 3.20 The central leaf through $x = (x_0, \lambda_0) \in A_g(k)$ is

$C(x) = \{ (x, \lambda) \in A_g(k) : (x, \lambda)\cdot [p^\infty] = (x_0, \lambda_0)\cdot [p^\infty] \}$

(Recall: $(x, \lambda)\cdot [p^\infty] = \lim_{n \to \infty} (x, \lambda)\cdot [p^n]$.)

This is a closed subset of $A_g$ of dim 0.

Fixed under degree-$\ell$ isogenies ($\ell \neq p$).
Def \text{3.25}  \hspace{1cm} \text{An isogeny leaf through}  \\
(X_0, \lambda_0) \in A_g(k) \text{ contains all}  \\
(X_0, \mu_0) \in A_g(k) \text{ isogenous to} (X_0, \lambda_0)  \\
\text{via an iterated } \alpha_p \text{-isogeny.}  \\

\text{This is a closed integral subscheme,}  \\
\text{for } (X_0, \lambda_0) \in A_g(k) \text{ of dim } \left\lfloor \frac{g^2}{4} \right\rfloor.

Remark. The theorem holds for any irr cpt \( V \subseteq W_{\xi}^0 \subseteq A_g \) of an open Newton stratum.

\[
\dim \text{ central leaf} = c_{\xi} \\
(> 0 \text{ whenever } \xi \neq \sigma)
\]
[Chai]

irreducible whenever \( \xi \neq \sigma \).

\[
\dim \text{ isogeny leaf} = 1/\Delta(\xi) - c_{\xi} = i_{\xi}
\]
Thm 3.25 (Ibukiyama - K - Hu)

For \( x \in \mathcal{G}(k), \quad x \not\equiv (x, 1) \)

\# \mathcal{E}(x) = 1 \iff \text{one of the following holds:}

- \( g = 1, \ p \in \{2, 3, 5, 7, 13\} \)
- \( g = 2, \ p \in \{2, 3\} \)
- \( g = 3, \ p = 2, \ a(X) > 2 \).

(In these cases, \( \sqrt{\cdot} \) = isogeny leaf, and \( x \) is determined by its \( p \)-divisible group.)