HEIGHTS PROBLEM SET 4

Below you will find some problems to work on for Week 4! There are three categories: beginner, intermediate and advanced.

Beginner problems

Question 1. Let $K = \mathbb{Q}(\alpha)$ be a number field. Let f be the minimal polynomial of α, and let p be a prime that does not divide the index $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$. Suppose f factors as

$$f(x) \equiv f_1(x)^{e_1} \dots f_r(x)^{e_r} \mod p,$$

where $f_i(x) \in \mathbb{Z}[x]$ such that $f_i(x) \mod p$ are pairwise distinct irreducible polynomials in $\mathbb{F}_p[x]$. Let $\mathfrak{p}_i := (p, f_i(\alpha))$ for each i. Verify that $\mathfrak{p}_i$ is a prime ideal.

Question 2. Let K be a number field and $\mathcal{O}_K$ be its ring of integers.

(1) Show that if I is a nonzero ideal of $\mathcal{O}_K$, then $I \cap \mathbb{Z}$ is a nonzero ideal of $\mathbb{Z}$. Use this to show that I has finite index in $\mathcal{O}_K$.
(2) Show that if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$, then $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$.
(3) Prove that every finite integral domain is a field. (Hint: To prove that a nonzero element α has a multiplicative inverse, consider the set $\{\alpha, \alpha^2, \dots\}$.)
(4) Combine the previous three parts to show that if $\mathfrak{p}$ is a nonzero prime ideal of $\mathcal{O}_K$, then $\mathfrak{p}$ is in fact a maximal ideal. If p is a generator for the ideal $\mathfrak{p} \cap \mathbb{Z}$, then $\mathcal{O}_K/\mathfrak{p}$ is a finite extension of the finite field $\mathbb{F}_p$.

Question 3. Let K be a number field and let p be a prime number that does not divide the index $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$. If $\mathfrak{p}_i$ is the prime ideal associated to the irreducible polynomial $f_i(x)$ appearing in the factorization of f modulo p, show that the inertial degree of $\mathfrak{p}_i$ is the degree of the polynomial f_i.

Question 4. Let $K = \mathbb{Q}(\sqrt{-1})$. Compute the relative height H_K of $P := [5, 6]$. Use this to compute $H(P)$.

Intermediate problems

Question 5. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$, where K is a number field.

(1) Show that $\mathfrak{p}^i \neq \mathfrak{p}^{i+1}$ for any integer i.
(2) Let $\alpha \in \mathfrak{p}^i \backslash \mathfrak{p}^{i+1}$. Show that the map of $\mathcal{O}_K$-modules $\mathcal{O}_K/\mathfrak{p} \to \mathfrak{p}^i/\mathfrak{p}^{i+1}$ induced by sending 1 to α is an isomorphism.
(3) Verify that the dimension of $\mathcal{O}_K/\mathfrak{p}^r$ as a $\mathbb{F}_p$ vector space is $rf(\mathfrak{p}|p)$.

Question 6. Assume that K is a number field.

(1) Show that every ideal of $\mathcal{O}_K$ is generated by at most two elements.
(2) Show that $\mathcal{O}_K$ is a PID if and only if it is a UFD.

Question 7. Prove that if $\alpha \in K$ for a number field K, then $H(\alpha) = H([\alpha\colon 1])$.

Question 8. Let $K/\mathbb{Q}$ be a finite Galois extension. Show that if $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ and $P = [x_0, \dots, x_n] \in \mathbb{P}^n(K)$. Then,

$$H_K(\sigma(P)) = H_K(P),$$

where $\sigma(P) = [\sigma(x_0), \dots, \sigma(x_n)]$.

Question 9. Show that the two different embeddings $K := \mathbb{Q}\left(\sqrt{2}\right) \to \mathbb{R}$ induce different topologies on K. (Hint: Can you construct a sequence of elements of K that converges to 0 in one topology but does not converge in the other?)

Advanced problems

Question 10. (Generalized Liouville's inequality). Let L/K be an extension of number fields and S be a finite set of primes in $\mathcal{O}_L$. Let α, β be elements of L with $\alpha \neq \beta$.

(a) Show that $H(\alpha - \beta) \leqslant 2H(\alpha)H(\beta)$.
(b) Show that $\prod_{\mathfrak{p}\in S} |\alpha|_{\mathfrak{P}} \leqslant H(\alpha)$.
(c) Show that

$$(2H(\alpha)H(\beta))^{-1} \geqslant \prod_{\mathfrak{p}\in S} |\alpha - \beta|_{\mathfrak{P}} \leqslant 2H(\alpha)H(\beta).$$

[Hint: For the lower bound use that $H(\gamma) = H(1/\gamma)$ for any $\gamma \in \overline{Q}$.]

Question 11. Prove that if $P \in \mathbb{P}^n(K)$ with homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$, where $x_i \in K$ for $i \in \{0, \ldots, n\}$ and one of the coordinates is equal to 1, then

$$H(P) \geqslant \left(\prod_{i=0}^{n} H(x_i)\right)^{1/n}.$$

Question 12. Prove the product formula for number fields: for $x \in K^*$ we have

$$\left(\prod_{\mathfrak{p}\in \mathrm{MSpec}(\mathcal{O}_K)} |x|_{\mathfrak{p}}\right)\left(\prod_{i=1}^{r} |\sigma_i(x)|_{\mathbb{R}}\right)\left(\prod_{j=1}^{s} |\tau_j(x)|_{\mathbb{C}}^2\right) = 1.$$

(Hint: Let $x \in \mathcal{O}_K\backslash\{0\}$. Compute the size of $\mathcal{O}_K/x\mathcal{O}_K$ in two ways: (1) Show that it equals the product of the terms coming from the Archimedean places. (2) Show that if $x\mathcal{O}_K = \mathfrak{p}_1^{e_1} \ldots \mathfrak{p}_r^{e_r}$ and $\mathfrak{p}_i \cap \mathbb{Z} = p_i\mathbb{Z}$ with $p_i > 0$, then $\#\mathcal{O}_K/x\mathcal{O}_K = \prod p_i^{e_i f_i}$). This is analogous to the proof of the product formula over $\mathbb{Q}$.