HEIGHTS PROBLEM SET 2

Below you will find some problems to work on for Week 2! There are three categories: beginner, intermediate and advanced.

Beginner problems

Question 1. Suppose that the minimal polynomial \( f \in \mathbb{Z}[x] \) of \( \alpha \) factors as
\[
f(x) = a_0x^n + \ldots + a_n = a_0(x - \alpha_1) \cdots (x - \alpha_n)
\]
over \( \mathbb{C} \). Then prove that for every \( i \) between 0 and \( n \), we have
\[
a_i/a_0 = (-1)^i \sum_{1 \leq s_1 < s_2 < \ldots < s_i \leq n} \alpha_{s_1} \alpha_{s_2} \cdots \alpha_{s_i}.
\]

Question 2. In this problem, you will show that \( H(\alpha^{-1}) = H(\alpha) \).
(a) If \( \alpha \) is a nonzero algebraic number with minimal polynomial \( f(x) := a_0x^n + a_1x^{n-1} + \ldots + a_n \), then verify that \( 1/\alpha \) is also an algebraic number with minimal polynomial
\[
f^{\text{rev}}(x) := x^nf(1/x) = a_0 + a_1x + \ldots + a_nx^n
\]
if \( a_n > 0 \), and minimal polynomial \(-f^{\text{rev}}(x)\) if \( a_n < 0 \).
(b) Describe the roots of \( f^{\text{rev}}(x) \) in terms of the roots of \( f(x) \).
(c) Show that \( H(\alpha^{-1}) = H(\alpha) \). \text{\textit{Hint: use Question 1.}}

Question 3. This question will introduce you to splitting fields and get you more comfortable computing with number fields. Recall the table from Padma’s notes:

<table>
<thead>
<tr>
<th>Algebraic number</th>
<th>Minimal polynomial</th>
<th>Number field</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a/b \in \mathbb{Q} ) &lt;br&gt;\text{gcd}(a, b) = 1, \ b &gt; 0 | ( bx - a ) | ( \mathbb{Q} ) | 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( i ) | ( x^2 + 1 ) | ( \mathbb{Q}(i) \cong \mathbb{Q}[x]/(x^2 + 1) ) | 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sqrt{2} + 1 ) | ( (x - 1)^2 - 2 ) | ( \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2) ) | 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sqrt{2} ) | ( x^2 - 2 ) | ( \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2) ) | 3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \zeta_p ), a primitive ( p)-th root of unity for a prime ( p ) | ( \varphi_p(x) := x^{p-1} ) | ( \mathbb{Q}(\zeta_p) \cong \mathbb{Q}[x]/(\varphi_p(x)) ) | ( p - 1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) For each of the rows of the table, do the following.
- Find all of the roots of the minimal polynomial over the number field. How many roots do you find?
- Factor the minimal polynomial over the number field.
(c) Answer the same questions for the polynomial \( f(x) := x^3 - 2 \) over \( S := \mathbb{Q}[x]/(x^6 - 108) \). You should only get linear factors. We call the number field \( S \) the splitting field of \( f(x) \): the smallest field extension of the base field over which \( f(x) \) \textit{splits} (decomposes into linear factors).

Intermediate problems

Question 4. Prove Gauss’ lemma: a polynomial \( f := a_0x^n + a_1x^{n-1} + \ldots + a_n \) in \( \mathbb{Z}[x] \) is irreducible if and only if it is irreducible in \( \mathbb{Q}[x] \) and \( \text{gcd}(a_0, \ldots, a_n) = 1 \).

Question 5. Prove that any irreducible polynomial of degree \( n \) in \( \mathbb{Q}[x] \) has \( n \) distinct roots in \( \mathbb{C} \).
Question 6. There is also a third definition of a height function $H_3$, in terms of the house $\hat{\alpha}$ and denominator $\text{den}$ of an algebraic number $\alpha$ (See also [Wal00]§ 3.4):

$$\hat{\alpha}(\alpha) := \sqrt[n]{\max_{j=1}^{\infty} |\alpha_j|}$$

$$\text{den}(\alpha) := \min\{D \in \mathbb{Z} : D > 0, \ D\alpha \text{ has a monic minimal polynomial in } \mathbb{Z}[x]\}$$

$$H_3(\alpha) := \text{den}(\alpha) \max\left(1, \hat{\alpha}(\alpha)\right).$$

Prove that $\text{den}(\alpha)$ is well-defined and divides the leading coefficient $a_0$ of the minimal polynomial $a_0x^n + \ldots + a_n$ of $\alpha$. Prove explicit inequalities relating $H(\alpha), H_2(\alpha)$ and $H_3(\alpha)$.

Question 7. Fix $m \geq 1$. Consider the polynomial $g$ defined by

$$g(x) := a_0^m (x - \alpha_1^m) \cdots (x - \alpha_n^m).$$

Show that $g(x) \in \mathbb{Z}[x]$ and that it is a power of the minimal polynomial of $\alpha^m$.

Question 8. Consider an algebraic number $\alpha$ with minimal polynomial $f(x) = a_0x^n + \ldots + a_n \in \mathbb{Z}[x]$, and conjugates $\alpha_1, \ldots, \alpha_n$. Let

$$\text{Disc}(f) = a_0^{2n-2} \prod_{i>j} (\alpha_i - \alpha_j)^2$$

be the discriminant of $f$. Show that

$$\frac{1}{n} \log |\text{Disc}(f)| \leq \log n + (2n - 2)h(\alpha).$$

Advanced problems

Question 9.

(a) Prove Liouville’s inequality, namely that if $\alpha$ is an algebraic irrational number of degree $n \geq 2$, then there is a constant $C$ (depending on $\alpha$), such that for any rational number $a/b$ with $b > 0$, we have

$$|\alpha - \frac{a}{b}| \geq \frac{C}{b^n}.$$

(Hint: Let $f$ be the minimal polynomial of $\alpha$. Combine a lower bound on the nonzero rational number $f(a/b)$ and an upper bound for $|f(\alpha) - f(a/b)|/(\alpha - (a/b))$ using the Mean Value Theorem.)

(b) A Liouville number is a real number $x$ with the property that for any integer $n$, there is a rational number $a/b$ with $b > 1$ such that

$$0 < |x - (a/b)| < 1/b^n.$$

Prove that Liouville numbers are transcendental and that Liouville’s constant $\sum_{k=1}^{\infty} \frac{1}{10^k}$ is a Liouville number.

References