MODEL THEORY PROBLEM SET 6

Beginner problems

Question 1: Let $F \models RCF$ and $A \subseteq F^n$ be semialgebraic. Show that the closure (in the Euclidean topology) of $F$ is semialgebraic.

Question 2: Let $x$ and $y$ be algebraically independent over $\mathbb{R}$. Show that $\mathbb{R}(x,y)$ is formally real and that we can find orderings $<_1$ and $<_2$ of $\mathbb{R}(x,y)$ such that $x<_1 y$ and $y<_2 x$.

Question 3: Let $F$ be a field. Show that $F$ is formally real if and only if for all $a_1, \ldots, a_n \in F$ we have that
\[ a_1^2 + \cdots + a_n^2 = 0 \implies a_1 = \cdots = a_n = 0. \]

Question 4: Let $\mathcal{L} = \{0, 1, +, -, \times, <, f\}$, where $f$ is a function symbol.
(a) Consider $\mathbb{R}$ as an $\mathcal{L}$-structure, where $f$ is interpreted as the sine function on $\mathbb{R}$: $\sin(x)$. Is the theory of $\mathbb{R}$ in this language o-minimal?
(b) Consider $\mathbb{R}$ as an $\mathcal{L}$-structure, but now interpret $f$ as the complex exponential function on $\mathbb{C}$ (so now $f$ is a function of two variables). Is the theory of $\mathbb{R}$ in this language o-minimal? \footnote{A very famous theorem of Wilkie shows that if we interpret $f$ as the real exponential function, then the theory of $\mathbb{R}$ is o-minimal.}

Question 5: Let $F \models RCF$ and $S \subseteq F^{m+n}$ be semialgebraic. For $\bar{a} \in F^m$, let $S_{\bar{a}} := \{ \bar{b} \in F^n : (\bar{a}, \bar{b}) \in S \}$. Show that the set
\[ \{ \bar{a} \in F^m : S_{\bar{a}} \text{ is open} \} \]
is semialgebraic.

Intermediate problems

Question 6: Let $F \models RCF$, $X \subseteq F^n$ closed and bounded, and $f : F^n \to F$ semialgebraic (i.e., the graph of $f$ is semialgebraic). Show that $f(X)$ is closed and bounded. (Hint: notice that this is true in $\mathbb{R}$, and transfer the result over to any other real closed field.)

Question 7: Let $F \models RCF$ and $f(\bar{X}) \in F(\bar{X}_1, \ldots, \bar{X}_n)$ be a rational function. We say that $f$ is positive semidefinite if $f(\bar{a}) \geq 0$ for all $\bar{a} \in F^n$. Show that if $f$ is positive semidefinite, then $f$ is a sum of squares of rational functions.

Question 8: Let $\mathcal{L}$ be a language that contains the the symbol $\prec$. Let $T$ be an o-minimal theory on this language and let $\mathcal{M}$ be a model of $T$. Suppose that $\phi(x)$ is an $\mathcal{L}$-formula which has only finitely many realizations in $\mathcal{M}$. Show that for every $m \in \mathcal{M}$ realizing $\phi(x)$ there is another $\mathcal{L}$-formula $\psi(x)$ such that the only realization of $\psi(x)$ is $m$.

Question 9: Let $\mathcal{L} = \{e, *, \prec\}$ and let $G$ be an ordered group (i.e. $G$ is a group and for all $x, y, z \in G$ we have that $x < y \implies x * z < y * z$). Considering $G$ as an $\mathcal{L}$-structure, let $T$ be the complete first-order theory of $G$ in the language $\mathcal{L}$. Suppose that $T$ is o-minimal.
(a) Let $X \subseteq G$ be a definable set (defined with parameters from $G$). Show that if $X$ is a subgroup of $G$, then either $X = \{0\}$ or $X = G$.
   Hint: First show that if $X \neq \{0\}$, then there is $h \in G$ such that $(-h, h) \subseteq X$.\footnote{A very famous theorem of Wilkie shows that if we interpret $f$ as the real exponential function, then the theory of $\mathbb{R}$ is o-minimal.}
(b) Show that $G$ is abelian.
Hint: Given $h \in G$, consider the definable (with parameter $h$) subgroup $C(h) := \{g \in G : g \ast h = h \ast g\}$.
(c) Show that $G$ is divisible, i.e. for every $g \in G$ and every positive integer $n$, there exists $h \in G$ such that $nh = g$.

**Question 10:** Let $\mathcal{L}$ be a language that contains the symbol $<$ and let $\mathcal{M}$ be an o-minimal $\mathcal{L}$-structure. Assume that the underlying ordered set of $\mathcal{M}$ is densely ordered. Show that for $a < b \in M$, if $f : [a.b] \to M$ is a definable continuous function, then $f$ assumes all values between $f(a)$ and $f(b)$.

**Advanced problems**

**Question 11:** Let $\mathcal{L} = \{0, 1, +, -, \times, \{f\}_{i \in I}\}$, where $\{f\}_{i \in I}$ denotes a set of function symbols. Suppose we interpret the $f_i$ on $\mathbb{R}$ in such a way that the theory of $\mathbb{R}$ in this language is o-minimal.
(a) Let $g : \mathbb{R} \to \mathbb{R}$ be a definable function and assume that $g^{-1}(x)$ is a finite set for all $x \in \mathbb{R}$. Show that there is a positive integer $N$ such that for all $x \in R$, $g^{-1}(x)$ has at most $N$ elements.
(b) Let $g : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a definable function and assume that $g^{-1}(\pi)$ is a finite set for all $\pi \in \mathbb{R}^n$. Show that there is a positive integer $N$ such that for all $\pi \in \mathbb{R}^n$, $g^{-1}(\pi)$ has at most $N$ elements.
Hint: Use cell-decomposition and induction.

**Question 12:** Let $\mathcal{L}$ be the language of ordered rings. Let $R$ be an ordered ring such that the complete first-order theory $T$ of $R$ in the language $\mathcal{L}$ is o-minimal. Show that $R$ is a real closed field.