MODEL THEORY PROBLEM SET 4

Beginner problems

Question 1: (a) Let $\mathcal{F}$ be a filter on I, and suppose that $X \notin \mathcal{F}$. Let $\mathcal{F}' = \{Y \subseteq I : \exists Z \in \mathcal{F},\ Z \backslash X \subseteq Y\}$. Prove that $\mathcal{F}'$ is a filter, $\mathcal{F} \subseteq \mathcal{F}'$, and $I \backslash X \in \mathcal{F}'$.

(b) Prove that every filter can be extended to an ultrafilter. (Hint: use Zorn's lemma.)

Question 2: Prove that in the definition of an ultraproduct, the interpretation of function and relation symbols does not depend on the choice of representative. That is, given a family of $\mathcal{L}$-structures $\{\mathcal{M}_i : i \in I\}$ and an ultrafilter $\mathcal{U}$ on I, prove that:

(a) if $f \in \mathcal{L}_{\mathcal{F}}$ is a function symbol and $g_1, \ldots, g_{n_f}, g'_1, \ldots, g'_{n_f} \in \prod_{i\in I} \mathcal{M}_i$ are such that $g_i \sim g'_i$ for $i = 1, \ldots, n_f$, then taking $g, g' \in \prod_{i\in I} \mathcal{M}_i$ to be the functions such that $g(i) = f^{\mathcal{M}_i}(g_1(i), \ldots, g_{n_f}(i))$ and $g'(i) = f^{\mathcal{M}_i}(g'_1(i), \ldots, g'_{n_f}(i))$, we have that $g \sim g'$; and

(b) if $R \in \mathcal{L}_{\mathcal{C}}$ is a relation symbol and $g_1, \ldots, g_{n_R}, g'_1, \ldots, g'_{n_R} \in \prod_{i\in I} \mathcal{M}_i$, then

$$\{i \in I : (g_1(i), \ldots, g_{n_R}(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{U} \iff \{i \in I : (g'_1(i), \ldots, g'_{n_R}(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{U}.$$

Question 3: (a) Let $\mathcal{L} = \{0, 1, +, \times, -\}$ be the language of rings, and let T be the theory of fields. Let $\phi(u)$ be the formula $\exists v(uv = 1)$. Find a quantifier-free formula $\psi(u)$ such that $T \models \forall u\, (\phi(u) \iff \psi(u))$.

(b) Let $\mathcal{L} = \{0, 1, +, \times, -, <\}$ be the language of ordered rings, and consider $\mathbb{R}$ as an $\mathcal{L}$-structure. Let T be the complete first-order theory of $\mathbb{R}$ in this language. Let $\phi(x)$ be the formula $\exists y(x = y^2)$. Find a quantifier-free formula $\psi(u)$ such that $T \models \forall u\, (\phi(u) \iff \psi(u))$.

Question 4: Use the compactness theorem to show that for every field F there is a field extension L such that there exists an element $t \in L$ which is transcendental over F.

Intermediate problems

Question 5: Let $\mathcal{L} = \{0, 1, +, \times, -\}$ be the language of rings, and consider $\mathbb{C}$ as an $\mathcal{L}$-structure. Let T be the complete first-order theory of $\mathbb{C}$ in this language. Let $\phi(a, b, c, d)$ be the formula

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \left(\bigwedge_{i=1}^{4} x_i^4 + a x_i^3 + b x_i^2 + c x_i + d = 0 \wedge \bigwedge_{i<j} x_i \neq x_j \right).$$

Find a quantifier-free formula $\psi(a, b, c, d)$ such that

$$T \models \forall a \forall b \forall c \forall d\, (\phi(a, b, c, d) \iff \psi(a, b, c, d)).$$

Question 6: Let I be an infinite set. Prove that any nonprincipal ultrafilter $\mathcal{U}$ on I extends the Fréchet filter.

Question 7: Let $\mathbb{P}$ be the set of prime numbers and let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{P}$. For each $p \in \mathbb{P}$, consider the cyclic group $\mathbb{Z}/p\mathbb{Z}$, and let $\mathcal{G}$ be the ultraproduct

$$\mathcal{G} = \left(\prod_{p\in\mathbb{P}} \mathbb{Z}/p\mathbb{Z} \right) / \, \mathcal{U}.$$

Then $\mathcal{G}$ is a group by Łoś' Theorem, and we let 0 denote the identity element of $\mathcal{G}$.

(a) Prove that $\mathcal{G}$ is torsion-free: for all $a \in \mathcal{G}$, if $a \neq 0$, then $na \neq 0$ for any nonzero $n \in \mathbb{N}$.

(b) Use Łoś' Theorem to conclude that the class of groups with a torsion element (that is, an element $a \neq 0$ with $na = 0$ for some nonzero $n \in \mathbb{N}$) is not axiomatizable.

Question 8: Let $\mathbb{P}$ be the set of prime numbers and let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{P}$. For each $p \in \mathbb{P}$, consider the finite field $\mathbb{F}_p$, and let F be the ultraproduct

$$F = \prod_{p \in \mathbb{P}} \mathbb{F}_p \,/\, \mathcal{U}.$$

(a) What can you say about the characteristic of F?
(b) Show that F has an algebraic extension of degree n for every n. Challenge: show that these extensions are unique.

Advanced problems

Question 9: Let $\mathcal{L} = \{0, 1, +, \times, -, <\}$ be the language of ordered rings, and consider $\mathbb{Q}$ as an $\mathcal{L}$-structure. Let T be the complete first-order theory of $\mathbb{Q}$ in this language. Let $\phi(x)$ be the formula $\exists y(x = y^2)$. Is it possible to find a quantifier-free formula $\psi(u)$ such that $T \models \forall u\,(\phi(u) \iff \psi(u))$?

Question 10: (a) Use Proposition 4.12 (Test for QE) to prove that DLO, the theory of dense linear orders without endpoints, has quantifier elimination.
(b) Use part (a) to show that $\mathbb{Z}$ is not a definable subset of $(\mathbb{Q}, <)$. Hint: show that any definable subset of $\mathbb{Q}$ is a finite union of points $\{a\}$ and open intervals (b, c).

Question 11: Let $f : \mathbb{R} \to \mathbb{R}$ be a function, let $\mathcal{R}$ be the structure $(\mathbb{R}, 0, 1, +, \times, -, <, f)$. Let $\mathcal{M} = (M, 0, 1, +, \times, -, <, f^{\mathcal{M}})$ be an elementary extension of $\mathcal{R}$ which contains infinitesimal elements (such an extension exists by the compactness theorem). Prove that for any $r \in \mathbb{R}$, the function f is continuous at r if and only if $f^{\mathcal{M}}(r + \epsilon) - f^{\mathcal{M}}(r)$ is infinitesimal for all infinitesimal $\epsilon \in M$.