MODEL THEORY PROBLEM SET 3

Beginner problems

Question 1: Prove Fact 3.1: if $\rho : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, then there is an $\mathcal{L}$-structure $\mathcal{N}'$ isomorphic to $\mathcal{N}$ such that $\mathcal{M} \preceq \mathcal{N}$

Question 2: Let $\mathcal{L} = \{E\}$, where E is a binary relation symbol. The $\mathcal{L}$-theory of (undirected) graphs is axiomatized as follows:

- $\forall x \neg(xEx)$;
- $\forall x \forall y(xEy \to yEx)$.

A graph $\mathcal{G} = (V, E)$ is said to be *connected* if for any vertices $a, b \in V$, either aEb or there are vertices $v_1, \ldots, v_n \in V$ with $aEv_1Ev_2E \cdots Ev_nEb$. Prove that the theory of connected graphs is not first order axiomatizable.

Question 3: Let $\mathcal{L} = \{0, 1, +, \times, <\}$ and consider $\mathbb{N}$ as an $\mathcal{L}$-structure. Recall that $\mathrm{Th}(\mathbb{N})$ is the set of all $\mathcal{L}$-sentences satisfied by $\mathbb{N}$, Show that there is a model $\mathcal{M} \models \mathrm{Th}(\mathbb{N})$ and $a \in M$ such that a is larger than every natural number.

Question 4: Let $\mathcal{L}_{or} := \{0, 1, +, \times, -, <\}$ (this is known as the language of *ordered rings*) and consider $\mathbb{R}$ as an $\mathcal{L}_{or}$-structure..

(a) Show that there is an elementary extension $\mathcal{M} \succeq \mathrm{Th}(\mathbb{R})$ such that there exists $m \in M$ satisfying $m > 0$ and $m < \frac{1}{n}$ for all positive integers n (such an m is called an *infinitesimal* element).

(b) Show that there is an elementary extension $\mathcal{N} \succeq \mathrm{Th}(\mathbb{R})$ such that there exists $n \in N$ satisfying $n > r$ for all real numbers $r \in \mathbb{R}$ (such an n is called an *infinite* element).

Intermediate problems

Question 5: Show that the theory of Abelian groups, where every element has order 2 is κ-categorical for all infinite κ. Show that it is not complete. Why does Vaught's Test not apply?

Question 6: Let $\mathcal{L} := \{0, +, \{f_q\}_{q \in \mathbb{Q}}\}$ be the language used in the first problem set to write down the axioms of $\mathbb{Q}$-vector spaces (here f_q is a function symbol in one variable that is interpreted as scalar multiplication by q). Let T be the theory given by these axioms. Show that T is complete.

Hint: Show that T is κ categorical, for $\kappa > \aleph_0$.

Question 7: Let $\mathcal{L} = \{<\}$, and consider $\mathbb{Z}$ as an $\mathcal{L}$-structure. Notice that $\mathrm{Th}(\mathbb{Z})$ extends the axioms for linear orders. Show that there is a model $\mathcal{M} \models \mathrm{Th}(\mathbb{Z})$ such that there is an order preserving embedding $\sigma : \mathbb{Q} \to \mathcal{M}$. (Hint: add constants for each element of $\mathbb{Q}$.)

Question 8: Zermelo–Fraenkel set theory with the axiom of choice (ZFC) is a first-order theory in the language $\mathcal{L} = \{\in\}$, where $\in$ is a binary relation symbol. Recall that ZFC asserts the existence of an uncountable set, namely the power set of ω. Prove Skolem's Paradox: that despite implying the existence of an uncountable set, ZFC has a countable model (assuming that it is consistent). Why is this not actually a contradiction? That is, why does it not follow that ZFC is inconsistent?

Advanced problems

Question 9: Let $\mathcal{L} = \{<\}$ and let DLO, the theory of dense linear orders without endpoints, be the $\mathcal{L}$-theory consisting of the following axioms:

- $<$ is a total order;
- $<$ is dense: for any elements $a < b$, there is an element c with $a < c < b$;
- $<$ does not have endpoints: for any element a there is are elements b and c with $b < a < c$.

Prove that DLO is $\aleph_0$-categorical. Conclude that any countable model of DLO is isomorphic to $(\mathbb{Q}, <)$. Hint: let $\mathcal{M}, \mathcal{N}$ be two countable models of DLO. Carefully construct an increasing sequence of partial maps from M to N with finite domains such that the union of these maps is an isomorphism.

Question 10: Let $\mathcal{L}$ be a language and let T be a complete theory in this language. Show that the following two conditions are equivalent:

(a) There exists a finite set of $\mathcal{L}$-sentences T_0 such that for every $\mathcal{L}$-structure $\mathcal{M}$ we have that $\mathcal{M} \models T$ if and only if $\mathcal{M} \models T_0$ (in this case we say that T is *finitely axiomatizable*).

(b) There exists a theory T' in the language $\mathcal{L}$ such that for every $\mathcal{L}$-structure $\mathcal{M}$ we have that $\mathcal{M}$ is not a model of T if and only if $\mathcal{M}$ is a model of T'.

Question 11: Suggest a language that one can use to write down the axioms of metric spaces. Let T be the theory give by these axioms. Can you find any models of T which are not metric spaces?