MODEL THEORY PROBLEM SET 1

Beginner problems

Question 1: Let $\mathcal{L}_r = \{0, 1, +, \times, -\}$ denote the language of rings. Consider the fields $\mathbb{R}$ and $\mathbb{C}$ as $\mathcal{L}_r$-structures. Find an $\mathcal{L}_r$-sentence ϕ so that
(a) $\mathbb{C} \models \phi$ and $\mathbb{R} \not\models \phi$.
(b) $\mathbb{R} \models \phi$ and $\mathbb{C} \not\models \phi$.

Question 2: In lectures we saw that the language $\mathcal{L}_g = \{*, e\}$ can be used to write down the axioms of groups. Find a language $\mathcal{L}_{\mathbb{Q}v.s.}$ which can be used to write out the axioms of $\mathbb{Q}$-vector spaces. Hint: you may need to use an infinite language.

Question 3: Consider the group of integers $\mathbb{Z}$ as an $\mathcal{L}_g$-structure. Prove that the set of even integers is definable. That is, find an $\mathcal{L}_g$-formula $\phi(x)$ such that

$$\mathbb{Z} \models \phi(a) \iff a \text{ is even}$$

for all $a \in \mathbb{Z}$.

Question 4: Let $\rho : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{L}$-isomorphism. Prove that

$$\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\rho(\bar{a}))$$

for all $\mathcal{L}$-formulas ϕ and all $\bar{a} \in M^n$.

Question 5: Let $\phi(x)$ be an $\mathcal{L}$-formula and n a natural number. Show that there is an $\mathcal{L}$-sentence ψ such that $\mathcal{M} \models \psi$ if and only if the definable set $Y = \{a \in M : \mathcal{M} \models \phi(a)\}$ has exactly n elements. What about expressing that Y has at most n elements? At least n elements? Infinitely many elements?

Intermediate problems

Question 6: Let $\mathcal{L}_r = \{0, 1, +, \times, -\}$ denote the language of rings. Consider the fields $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$, which are $\mathcal{L}_r$-structures.
(a) Are $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$ isomorphic as $\mathcal{L}_r$-structures?
(b) Is there an $\mathcal{L}_r$-sentence which is true in one of these fields but not on the other?

Question 7: A **quantifier free** $\mathcal{L}$-formula is an $\mathcal{L}$-formula which does not contain any instances of $\forall$ or $\exists$. Prove that every $\mathcal{L}$-formula ϕ is equivalent to an $\mathcal{L}$-formula of the form

$$Q_1 x_1 \cdots Q_n x_n \psi,$$

where ψ is a quantifier free $\mathcal{L}$-formula and each Q_i is either $\forall$ or $\exists$.

Question 8: An **existential** $\mathcal{L}$-formula is an $\mathcal{L}$-formula of the form $\exists x_1 \cdots \exists x_n \psi(x_1, \ldots, x_n)$, where ψ is a quantifier free $\mathcal{L}$-formula. Likewise, a **universal** $\mathcal{L}$-formula has the form $\forall x_1 \cdots \forall x_n \psi(x_1, \ldots, x_n)$, where ψ is quantifier free. Let $\mathcal{M}$ be an $\mathcal{L}$-substructure of $\mathcal{N}$. Prove that
(a) If ϕ is an existential $\mathcal{L}$-formula, then $\mathcal{M} \models \phi(\bar{a}) \implies \mathcal{N} \models \phi(\bar{a})$ for all $\bar{a} \in M^k$.
(b) If ϕ is a universal $\mathcal{L}$-formula, then $\mathcal{N} \models \phi(\bar{a}) \implies \mathcal{M} \models \phi(\bar{a})$ for all $\bar{a} \in M^k$.

Question 9: Let $\mathcal{L}$ be a finite language and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Prove that there is a sentence ϕ such that $\mathcal{N} \models \phi$ if and only if $\mathcal{M} \cong \mathcal{N}$.

Advanced problems

Question 10: Let $\mathcal{L} = \{0, 1, +, \times, -, \exp\}$ denote the language of exponential rings. Consider the exponential field $(\mathbb{C}, \exp)$ as an $\mathcal{L}$-structure. Prove that the set of integers $\mathbb{Z} \subseteq \mathbb{C}$ is $\mathcal{L}$-definable.

Question 11: Consider the language $\mathcal{L} := \{0, 1, +, \times, -, <, f\}$, where f denotes a function symbol in one variable. We can think of $\mathbb{R}$ as an $\mathcal{L}$-structure by choosing a function $F : \mathbb{R} \to \mathbb{R}$, where f is interpreted as F and the other symbols have their usual interpretations.

(a) Write an $\mathcal{L}$-sentence which says that $\lim_{x\to 0} F(x) = 1$.

(b) Write a sentence saying that F is continuous on $\mathbb{R}$.

Question 12: Given a sentence ϕ, the *spectrum of* ϕ is the set of natural numbers n such that there is $\mathcal{M} \models \phi$ with $|M| = n$.

(a) Let $\mathcal{L} = \{E\}$, where E is a binary relation. Write down a sentence ϕ in this language that expresses that E is an equivalence relation where each equivalence class has exactly 2 elements. Prove that the spectrum of ϕ is the set of positive even integers.

(b) Find a language $\mathcal{L}$ and an $\mathcal{L}$-sentence ϕ such that the spectrum of ϕ is $\{n^2 : n \in \mathbb{N}, n > 0\}$.

(c) Find a language $\mathcal{L}$ and an $\mathcal{L}$-sentence ϕ such that the spectrum of ϕ is $\{p^n : p \text{ is prime}, n > 0\}$.