INTRODUCTION TO MODEL THEORY WITH APPLICATIONS

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3. The first few big theorems

In this lecture we will look at the first few major theorems in basic model theory. In particular we introduce the Compactness Theorem one most fundamental result in model theory. We will delay its proof for the next lecture and instead focus on the consequences (or application) of this theorem.

3.1. **More on Elementary Substructures.** Let $\mathcal{L}$ be a fixed language. In the second lecture we introduced the concepts of elementary embeddings and elementary substructures. We recall here the definition for convenience.

Definition 2.14. Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. We say that an embedding $\rho : \mathcal{M} \to \mathcal{N}$ is an **elementary embedding** if for all $\mathcal{L}$-formulas $\phi(x_1, ..., x_n)$ and and all $\overline{a} \in M^n$ we have that $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(\rho(\overline{a}))$.
If $\mathcal{M} \subseteq \mathcal{N}$ and the inclusion map is also elementary, we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ and that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$. We write $\mathcal{M} \preceq \mathcal{N}$.

It turns out that the difference between an elementary substructure and elementary embedding is not too important:

Fact 3.1. [1] *If $\rho : \mathcal{M} \to \mathcal{N}$ is an elementary embedding, then there is an $\mathcal{L}$-structure $\mathcal{N}'$ isomorphic to $\mathcal{N}$ such that $\mathcal{M} \preceq \mathcal{N}'$.*

We will use this fact several times in this lecture. The following gives a criterion to check whether a given substructure is elementary.

Proposition 3.2 (Tarski-Vaught Test)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures such that $\mathcal{M} \subseteq \mathcal{N}$. Then the following is equivalent*

1. $\mathcal{M} \preceq \mathcal{N}$
2. *For any $\mathcal{L}_M$-formula $\phi(x)$ in one free variable, if $\mathcal{N}_M \models \exists x \phi(x)$ then there is $b \in M$ such that $\mathcal{N}_M \models \phi(b)$.*

Proof. (1)$\Rightarrow$(2) Suppose $\mathcal{M} \preceq \mathcal{N}$ and let $\phi(x)$ be an $\mathcal{L}_M$-formula in one free variable. Assume $\mathcal{N}_M \models \exists x \phi(x)$. Since $\mathcal{M} \equiv_M \mathcal{N}$, we have that $\mathcal{M}_M \models \exists x \phi(x)$. So by definition there is $b \in M$ such that $\mathcal{M}_M \models \phi(b)$. Since $\mathcal{M} \preceq \mathcal{N}$, the result follows.
(2)$\Rightarrow$(1) Assume the statement (2) holds. Let $\phi(y_1, ..., y_n)$ be a $\mathcal{L}$-formula, and $\overline{a} \in M^n$. We need to show that $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(\overline{a})$. Using Proposition 2.13, we know that this holds for quantifier free formulas. Using the usual arguments, we only need to prove by induction that it holds for the case when $\phi(\overline{y})$ is

[1] You will be ask to prove it in the problem session.

of the form $\exists x\psi(\overline{y},x)$ given that it already holds for $\psi(\overline{y},x)$. If $\mathcal{M}\models\phi(\overline{a})$ then there is $b\in M$ such that $\mathcal{M}\models\psi(\overline{a},b)$. By induction $\mathcal{N}\models\psi(\overline{a},b)$ and hence $\mathcal{N}\models\phi(\overline{a})$. On the other hand, assume that $\mathcal{N}\models\phi(\overline{a})$. By simply noticing that $\psi(\overline{a},x)$ is an $\mathcal{L}_M$ formula, we see that we can apply (2) to $\mathcal{N}\models\exists x\psi(\overline{a},x)$ and obtain an $b\in M$ such that $\mathcal{N}\models\psi(\overline{a},b)$. By induction $\mathcal{M}\models\psi(\overline{a},b)$ and so we get that $\mathcal{M}\models\phi(\overline{a})$ as required. $\square$

Remark 3.3. Note that by inspecting the proof of (2)⇒(1) carefully, we see that the statement (2) is only used in one direction. So together with Proposition 2.13 and the relevant arguments in the proof, we also obtain: Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures such that $\mathcal{M}\subseteq\mathcal{N}$. If $\mathcal{M}\models\phi(\overline{a})$ then $\mathcal{N}\models\phi(\overline{a})$ for any existential $\mathcal{L}$-formula $\phi(\overline{y}):=\exists x\psi(\overline{y},x)$ with $\psi(\overline{y},x)$ quantifier free and any $\overline{a}\in M^n$.

The Tarski-Vaught Test also holds for elementary embedding and we leave it for the reader to verify this fact. The following is believed to be the first Theorem in model theory. Recall that $|\mathcal{L}|$ is the cardinality of the set of all the symbols in $\mathcal{L}$.

Theorem 3.4 (Downward Löwenheim-Skolem)**.** *Suppose $\mathcal{N}$ is an $\mathcal{L}$-structure and $A\subseteq N$. Then there exists an elementary substructure of $\mathcal{M}\preceq\mathcal{N}$ such that $A\subseteq M$ and $|M|\leq max\{|A|,|\mathcal{L}|,\aleph_0\}$.*

In particular, if $\mathcal{L}$ is countable then every $\mathcal{L}$-structure has a countable elementary substructure.

Proof. For each $\mathcal{L}$-formula $\phi(\overline{y},x)$ we define a function $f_\phi:N^n\to N$ as follows:
for $\overline{a}\in N^n$

- If $\mathcal{N}\models\exists x\phi(\overline{a},x)$, choose any $b\in N$ such that $\mathcal{N}\models\phi(\overline{a},b)$ and set $f_\phi(\overline{a})=b$,
- if $\mathcal{N}\models\neg\exists x\phi(\overline{a},x)$, then set $f_\phi(\overline{a})=b$ for some arbitrary $b\in N$.

We let $M_0=A$ and define

$$M_{i+1}=M_i\cup\{f_\phi(\overline{a}):\phi(\overline{y},x)\text{ is an }\mathcal{L}\text{-formula and }\overline{a}\in M_i\}.$$

We also set $M=\bigcup M_i$.

Now notice that if $\phi(y,x)$ is of the form $y=c$ where $c\in L_{\mathcal{C}}$ is a constant symbol, then $f_\phi=c^{\mathcal{N}}$. Similarly, if $\phi(\overline{y},x)$ is of the form $f(\overline{y})=x$ where $f\in L_{\mathcal{F}}$ is a function symbol, then $f_\phi=f^{\mathcal{N}}$. In particular, for any $f\in L_{\mathcal{F}}$ and tuple $\overline{a}\in M^{n_f}$, since $\overline{a}\in M_i^{n_f}$ for some large enough i, we have that $f^{\mathcal{N}}(\overline{a})=f_\phi(\overline{a})\in M_{i+1}\subset M$. Hence, M contains all the constants of $\mathcal{N}$ and is preserved by all the functions in $\mathcal{N}$. As a result, M is the universe of a substructure $\mathcal{M}$ of $\mathcal{N}$.

We claim that $\mathcal{M}\preceq\mathcal{N}$. Indeed let $\phi(\overline{y},x)$ be an $\mathcal{L}$-formula and let $\overline{a}\in M^n$ be such that $\mathcal{N}\models\exists x\phi(\overline{a},x)$. Then $\mathcal{N}\models\phi(\overline{a},b)$ for $b=f_\phi(\overline{a})\in M$. By the Tarski-Vaught test, $\mathcal{M}\preceq\mathcal{N}$.

Finally, let $\kappa=max\{|A|,|\mathcal{L}|,\aleph_0\}$. We must show that $|M|\leq\kappa$. First observe that a formula $\phi(\overline{y},x)$ is a finite string from a set of symbols of size $|\mathcal{L}|+\aleph_0\leq\kappa$ and since κ is infinite, there can only be at most κ many formulas. If $|M_i|\leq\kappa$, then we claim that $|M_{i+1}|\leq\kappa$. Indeed, since $|M_i|\leq\kappa$, there can only be at most κ-many finite tuples $\overline{a}$ from M_i. Hence the set of all pairs $(\phi(\overline{y},x),\overline{a})$ (which is in bijection with the set of all $f_\phi(\overline{a})$ added to M_{i+1}) also has cardinality at most κ. So $|M_{i+1}| \leq \kappa$. By induction $|M_i| \leq \kappa$ for all i and since $M = \bigcup M_i$, we get that $|M| \leq \kappa$ as required. □

3.2. **The Compactness Theorem and its consequences.** We now state the Compactness Theorem, one of the most fundamental result in model theory. We will delay its proof for the next lecture. Instead we will focus on its many consequences.

Theorem 3.5 (Compactness Theorem)**.** *Suppose T is an $\mathcal{L}$-theory. T is consistent if and only if every finite subset of T is consistent.*

The right to left assertion is, of course, the key aspect of the theorem. It states that to prove that a theory is consistent, all we have to do is check whether an arbitrary finite subset is consistent. Let us illustrate its use in several applications. The following notion will be useful

Definition 3.6. Let $\mathcal{L} \subseteq \mathcal{L}'$ be languages. Let $\mathcal{M}'$ be an $\mathcal{L}'$-structure. The **reduct** of $\mathcal{M}'$ to $\mathcal{L}$ is the $\mathcal{L}$-structure $\mathcal{M}$ whose universe is the same as that of $\mathcal{M}'$ and such that interpretation in $\mathcal{M}$ of any symbol in $\mathcal{L}$ is the same as the interpretation in $\mathcal{M}'$.

Of course we have already seen the example of $(\mathbb{R}, +, \times, -, 0, 1)$ which is the reduct of $(\mathbb{R}, +, \times, -, 0, 1, <)$ to $\mathcal{L}_r$.

Proposition 3.7. *If an $\mathcal{L}$-theory T has arbitrarily large finite models. Then T has an infinite model.*

Proof. Let ϕ_n be the setentences

$$\phi_n := \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j,$$

and let T' be the new theory $T' = T \cup \{\phi_n : n \in \mathbb{N}_{>0}\}$. Then we have that any model of T' is an infinite model of T. In other words all we have to do is show that T' is consistent. By the Compactness Theorem all we have to show is that every finite subset of T' is consistent. So let Δ be a finite subset of T'. We have that for some $N > 0$, Δ is a subset of $T \cup \{\phi_1, \ldots, \phi_N\}$. Our assumption is that there is a model $\mathcal{M} \models T$ such that $|M| \geq N$, that is $\mathcal{M} \models \Delta$. □

Proposition 3.8. *The class of torsion groups is not first order axiomatizable.*

Proof. Recall that $\mathcal{L}_g = \{*, e\}$ is the language of groups and T_g the theory of groups. A group $G \models T_g$ is torsion if for all $h \in G$ we have that

$$h^n = \underbrace{h * \cdots * h}_{n-\text{times}} = e$$

for some $n \in \mathbb{N}$. Notice that $\forall h \exists n (h^n = e)$ is not a first order formula since we are only allowed to quantify over variables (i.e. ultimately over elements of the universe of structures).

Assume for contradiction that there is a set of $\mathcal{L}_g$-sentences S so that $T_{tor} = T_g \cup S$ axiomatizes the class of torsion groups. Let $\mathcal{L} = \mathcal{L}_g \cup \{c\}$ be the language which extend $\mathcal{L}_g$ by adding a new constant symbol c and consider the $\mathcal{L}$-theory

$$T' = T_{tor} \cup \{c^n \neq 0 : n \in \mathbb{N}_{>0}\}$$

which extend T_{tor}. It is not hard to see that T' is consistent. Indeed, if Δ is a finite subset of T' then for some $m > 0$ we have that Δ is contained in $T_{tor} \cup \{c^n \neq 0 : n \in \{1, \ldots, m\}\}$. Let $G_m = (\{0, 1, \ldots, m-1\}, +_m, 0)$ be the $\mathcal{L}_g$-structure where $+_m$ is addition modulo m. We of course know that $G_m \models T_{tor}$ (this is simply the additive group of the integers modulo m). However, we can make G_m into an $\mathcal{L}$-structure by interpreting c as 1. So indeed $G_m \models T_{tor} \cup \{c^n \neq 0 : n \in \{1, \ldots, m-1\}\}$ and hence Δ is consistent. By the Compactness Theorem, T' is consistent. If $G' \models T'$ then we can consider the its reduct G to $\mathcal{L}_g$ which satisfies $G \models T_{tor}$, that is G is torsion. This impossible since the interpretation of $c \in G'$ give that c is not torsion in G. So we get a contraction. $\square$

We now look at an example (of an application) of a different flavor. Let $\mathcal{Q} = (\mathbb{Q}^{alg}, +, \times, -, 0, 1)$. A natural questions to ask is whether all the elements in a model of $Th(\mathcal{Q})$ (or $Th(\mathcal{Q}_{\mathbb{Q}^{alg}})$) must be algebraic over $\mathbb{Q}$. That is, whether the algebraicity of $\mathbb{Q}^{alg}$ is part of the $Th(\mathcal{Q}_{\mathbb{Q}^{alg}})$. We show that it is not the case:

Proposition 3.9. *$\mathcal{Q}$ has an elementary extension which contains a transcendental element.*

Proof. For simplicity let us write $\mathcal{L} = L_r$ and $T = Th(\mathcal{Q}_{\mathbb{Q}^{alg}})$. Let $\mathcal{L}' = \mathcal{L}_{\mathbb{Q}^{alg}} \cup \{c\}$ where c is a new constant symbol. We work with the L'-theory

$$T' = T \cup \{F(c) \neq 0 : F \in \mathbb{Q}[X] \setminus \mathbb{Q}\}.$$

If Δ is a finite subset of T', then for some $N > 0$, it is contained in the set $T \cup \{F(c) \neq 0 : F \in \mathbb{Q}[X] \text{ and } deg(F) \leq N\}$. So in particular $\mathcal{Q}$ itself is a model of Δ by interpreting c as an algebraic number whose minimal polynomial is of degree $> N$. By the Compactness Theorem T' is consistent. Let $\mathcal{M}' \models T'$ and let $\mathcal{M}$ be the reduct of $\mathcal{M}'$ to $\mathcal{L}$. Notice that we have shown that $\mathcal{M}_{\mathbb{Q}^{alg}} \models Th(\mathcal{Q}_{\mathbb{Q}^{alg}})$. Let $\rho : \mathbb{Q}^{alg} \to M$ be the map defined by $\rho(q) = c_q^{\mathcal{M}_{\mathbb{Q}^{alg}}}$ where $q \in \mathbb{Q}^{alg}$ and c_q is the constant symbol in $\mathcal{L}_{\mathbb{Q}^{alg}}$ for q. Since $\mathcal{M}_{\mathbb{Q}^{alg}} \models Th(\mathcal{Q}_{\mathbb{Q}^{alg}})$, we have that $\mathcal{Q} \models \phi(\overline{a})$ if and only if $\mathcal{M} \models \phi(\rho(\overline{a}))$ for every $\mathcal{L}$-formula $\phi(\overline{x})$ and every $\overline{a}$ tuple from $\mathbb{Q}^{alg}$. That is $\rho : \mathcal{Q} \to \mathcal{M}$ is an elementary embedding. Finally, the interpretation of c in $\mathcal{M}'$ gives a transcendental element of $\mathcal{M}$. $\square$

Let us look at a more theoretical consequences of the Compactness Theorem.

Theorem 3.10 (Upward Löwenheim-Skolem)**.** *Suppose $\mathcal{M}$ is an infinite $\mathcal{L}$-structure and let $\kappa \geq max\{|M|, |\mathcal{L}|\}$. Then there exists an elementary extension of $\mathcal{M}$ of cardinality κ.*

In particular, if $\mathcal{L}$ is countable and T is an $\mathcal{L}$-theory with an infinite model, then T has models of any infinite cardinality

Proof. Let $\kappa \geq max\{|M|, |\mathcal{L}|\}$ and define $\mathcal{L}' := \mathcal{L}_M \cup \{c_\alpha : \alpha < \kappa\}$ where the c_αs are new constants symbols. Note that $|\mathcal{L}'| = \kappa$. Consider the $\mathcal{L}'$-theory

$$T' := Th(\mathcal{M}_M) \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \kappa\}.$$

It is not difficult to see that $\mathcal{M}$ itself is a model of any finite subset of T' by interpreting the c_αs which appear as some distinct elements (this is possible since $|M|$ is infinite). Hence by the compactness theorem, T' is consistent.

Let $\mathcal{N}' \models T'$, then its reduct $\mathcal{N}$ to $\mathcal{L}$ is such that $\mathcal{N} \models Th(\mathcal{M})$ and since $\mathcal{N}_M \models Th(\mathcal{M}_M)$, we can proceed as in the proof of Proposition 3.9 to show that $\mathcal{M} \preceq \mathcal{N}$. By construction $\mathcal{N}$ has at least κ many elements. Applying the Downward Löwenheim-Skolem Theorem to any $A \subseteq N$ that contains M and such that $|A| = \kappa$, one can show that there is an $\mathcal{N}_1$ of cardinality exactly κ such that $\mathcal{M} \preceq \mathcal{N}_1 \preceq \mathcal{N}$. □

Definition 3.11. Suppose that T is an $\mathcal{L}$-theory. We say that T is **κ-categorical** if all models of T of cardinality κ are isomorphic.

We can now give a criterion for proving that a consistent theory is complete.

Corollary 3.12 (Vaught's Test)**.** *Suppose T is an $\mathcal{L}$-theory with only infinite models. If T is κ-categorical for some infinite cardinal κ, then T is complete.*

Proof. Suppose T is κ-categorical. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models of T. We need to show that they are elementarily equivalent (using Theorem 2.10). Using the Upward and/or Downward Löwenheim-Skolem Theorem we obtain $\mathcal{M}_1', \mathcal{M}_2' \models T$ of cardinality κ such that $\mathcal{M}_1 \preceq \mathcal{M}_1'$ and $\mathcal{M}_2 \preceq \mathcal{M}_2'$. Using our assumption that T is κ-categorical and the fact that elementary embeddings (and isomorphisms) are elementarily equivalent (Theorem 2.16), we get $\mathcal{M}_1 \equiv \underbrace{\mathcal{M}_1' \equiv \mathcal{M}_2'}_{\text{via } \cong} \equiv \mathcal{M}_2$. □

We illustrate the use of Vaught's Test in the following examples

3.3. **Some complete theories.**

Example 3.13. Consider $\mathcal{L}_\emptyset = \emptyset$ the language of pure sets and $T_\infty = \{\phi_n : n \in \mathbb{N}_{>0}\}$, where

$$\phi_n := \exists x_1 \dots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j,$$

the theory of infinite sets. It is not hard to see that T_∞ is κ-categorical for all cardinals κ. Hence by Vaught's Test, T_∞ is complete.

Example 3.14. Let $\mathcal{L}_g = \{+, 0\}$ be the language of groups written additively. Let us denote by T_{ab} the theory T_g (of groups) together with the following sentences

$$\exists x (x \neq 0)$$

$$\forall x \forall y (x + y = y + x)$$

and for each $n \in \mathbb{N}_{>0}$ the sentences[2]

$$\forall x(x \neq 0 \to (nx \neq 0)$$
$$\forall y \exists x(nx = y).$$

The theory T_{ab} is called the theory of torsion-free divisible Abelian groups.

Proposition 3.15. *The theory T_{ab} is κ-categorical for all uncountable cardinal κ and hence complete.*

Proof. Notice first that if V is a vector space over $\mathbb{Q}$, then the underlying group of V is a model of T_{ab}. We claim that every model of T essentially arises this way. To see this, let us assume we have $G \models T$. For any $y \in G$ and $n \in \mathbb{N}_{>0}$, the axioms tell us that we can find $x \in G$ such that $nx = y$. It is not hard to see that x is the unique element of G with such property. Indeed, if $z \in G$ is such that $nz = y$, then we get that $n(x - z) = 0$. Using the axioms it must be that $x - z = 0$, i.e. $x = z$. We denote ths element x by y/n. So G can now be viewed as a vectors space over $\mathbb{Q}$ with the scalar multiplication defined as $\frac{m}{n}y = m(y/n)$.

We are now ready to go back to the proof of the proposition. First notice that in general for a $\mathbb{Q}$-vector space V, if V has dimension α, then $|V| = max\{\alpha, \aleph_0\}$. Consequently, if $G \models T_{ab}$ is such that $|G| = \kappa > \aleph_0$, then G as a $\mathbb{Q}$-vector space has dimension κ. But two vectors spaces over $\mathbb{Q}$ are isomorphic if and only if they have the same dimension. So any two models of T_{ab} of cardinality $\kappa > \aleph_0$ will have the same dimension κ and are hence isomorphic. So T_{ab} is κ-categorical for all uncountable cardinal κ and by Vaught's Test, T_{ab} is also complete. □

Remark 3.16. Notice that the existence of the countable models $\{\mathbb{Q}^n : n \in \mathbb{N}_{>0}\}$ of T_{ab} implies that T_{ab} is not $\aleph_0$-categorical.

[2]We will use the usual shorthand notation $nx = \underbrace{x + \cdots + x}_{n-\text{times}}$.