AWS 2023: SPECIAL POINT PROBLEMS

A large part of the subject of 'unlikely intersections' is 'special point problems', which involve the study of a distinguished set of points and potential algebraic relations between them (or in more fancy language, how they interact with the Zariski topology). Though it is possible to present the subject in quite a general and formal language, this field largely exists within a family of concrete examples which we shall aim to understand: Roots of unity, Torsion points in Abelian varieties, and CM points in Shimura varieties. Some topics we will cover are

- Lower bounds for Galois orbits of special points
- Upper bounds for heights of special points
- Class groups of number fields and Tori
- Abelian Varieties and the Masser-Wustholz inequality

A rough outline of the lectures (to be made more concrete later) is as follows:

(1) **Lecture 1: Roots of unity**
We shall introduce Lang's conjecture (known also as the multiplicative manin-munford conjecture) and give several approaches to it, some giving much more powerful equidistribution statements. This will cover the Galois theory of roots of unity, the 'Bezout method' of Edixhoven, Klingler, Ullmo, Yafaev and others, and other approaches. The benefit of this setting is that it serves as a fantastic 'trial run' for various approaches which then generalize to various extents in the other settings.

(2) **Lecture 2: Elliptic Curves, Galois Representations and Class Groups**
Here we shall discuss Elliptic curves with Complex multiplication and the resulting Andre-Oort conjecture. We shall relate them to class groups of quadratic fields and the Brauer-Siegel formula, and explain how the approaches from Lecture 1 generalize (or don't). We shall also explain how the Manin-mumford conjecture in this setting is related to Galois representations on the Tate-Module and various approaches to showing largeness of the Galois orbit.

(3) **Lecture 3: Moduli spaces of Abelian Varietes and the Masser-Wüstholz inequality**
We shall explain the challenges that show up when studying Abelian varieties, and how to overcome them. We shall present the Masser-Wüstholz inequality, and explain how it can be used to establish lower bounds for Galois orbits of CM abelian varieties.

(4) **Lecture 4: Generalizations: mixed Shimura varieties and Hodge theory**
We will explain how all the special cases studied thus far can be understood in a uniform way in the context of Mixed Shimura Varieties, and explain the main ideas in generalizing the Andre-Oort conjecture to this language. In a more general direction, we may view all these through the lens of Hodge theory and Hodge classes. It seems that the main obstacle to handling this setting in general is intimately related to the absolute Hodge conjecture.

1. PROJECTS

1.1. **Independence of CM points in Abelian Varieties.** Generalize the results about CM points [5, 3] mapping to linearly dependant points in Elliptic curves to Abelian varieties.

1.2. **Obtaining/effectivizing explicit exponents for Galois lower bounds.** Though a polynomial lower bound for Galois orbits of CM abelian varieties is known[7], an explicit exponent has not been worked out or at all optimized. This would involve understanding the various ingredients in the proof, and attempting to trace through an explicit exponent. The methods of [2] might be relevant, and related is [4].

If this goes very well, a potential direction for this project would be to try and classify all CM abelian varieties of dimension g defined over $\mathbb{Q}$, in an analogous way to the class number one problem.

1.3. **Good reduction of CM points.** One of the central properties of CM abelian varieties is that they have good reduction everywhere. For arbitrary Shimura varieties one may try to formulate the same question, except integral canonical models are far from known at all places. Nontheless, one may ask if there there is some integral model for which this is true?

This question comes up in [1] but is sidestepped for 'bad primes'. For the abelian variety case useful background reading could be the paper of Serre and Tate [6] and the theory of Complex Multiplication. This project has much relevance to, but little overlap with, the lecture material, and will rely on p-adic geometry, Galois theory, and some comfort with abstract Shimura Varieties. This is definitely the most difficult and speculative project.

References

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