**Point-counting and applications**

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Overview of lectures for Arizona Winter School 2023

**Lecture 1.** The basic point-counting result (for "definable sets in an o-minimal structure", but deferring a proper discussion of this notion) and its simplest application to an "unlikely intersection" problem: describing the distribution of torsion points on a subvariety of $(\mathbb{C}^\times)^n$, a problem also known as "Multiplicative Manin-Mumford".

**Lecture 2.** An introduction to definable sets in o-minimal structures, examples, and refinements of point-counting to count algebraic points of bounded degree. This encounters the situation when the "basic" statement can become trivial, but the proof of the counting theorem still yields a useful statement. This will be needed in the application in Lecture 3.

**Lecture 3.** On unlikely intersections for a curve in $Y(1)^n$. This is a true "unlikely intersection" problem, rather than a "special point" problem. We will go through the proof (of a partial result) emphasizing the arithmetic aspects, functional transcendence, point-counting, and further issues where o-minimality plays a role.

**Lecture 4.** In the lest lecture we will describe further applications and problems.

Projects

**Project 1.** To consider some variants and special cases of the result on unlikely intersections of an *asymmetric curve* in $Y(1)^n$ (theorem of Habegger-Pila). This would mainly involve getting good/explicit height bounds for the unlikely intersections in suitable settings.

**Project 2.** To consider generalizations/variants of some results of Jones-Thomas-Wilkie, Wilkie, Jones-Qiu on "integer-valued definable functions". For entire (instead of definable) functions this is a classic topic going back to Pólya (1915) and many variants have been studied. It involves an interplay of complex analysis in o-minimal structures with arithmetic properties.