O-minimality. 1

$(M, \ldots)$

$(\mathbb{C}, \times, +, 0, 1)$

$(\mathbb{R}, \times, +, 0, 1)$

$(\mathbb{R}, <, \times, +, 0, 1)$

$\underline{\underline{(\mathbb{R}, <)}}$

$(\mathbb{R}, <, \times, +, 0, 1, \underset{\mathbb{R}\to\mathbb{R}}{\exp})$

$(\mathbb{C}, \times, +, 0, 1, \underset{\mathbb{C}\to\mathbb{C}}{\exp})$

Definable set with parameters in $(M, \ldots)$

$A \subset M^n$

$A = \{(x_1, \ldots, x_n) \in M^n : \phi(\vec{x})\text{-holds}\}$

$\phi \in \mathrm{Form}(\mathcal{L}, \{c_m\})$

any $m \in M$

In o-minimality definability always with parameters.

minimal structure.

$(\mathbb{C}, \times, +, 0, 1)$.

$A \subset \mathbb{C}$ definable

<u>finite</u> or co<u>finite</u>

$(\mathbb{R}, <, \ldots)$ not minimal

Definition

A structure $(M, <, \ldots)$ expanding a dense linear order without endpoints is **O-minimal** if the definable subsets of M are just finite unions of pts and open intervals

i.e. just the definable sets in $(M, <)$

$(M, <, +, \times, 0, 1, \ldots)$

Van den Dries

$(\mathbb{R}, <, 0, 1, \times, +, \underset{\mathbb{R}\to\mathbb{R}}{\exp})$.

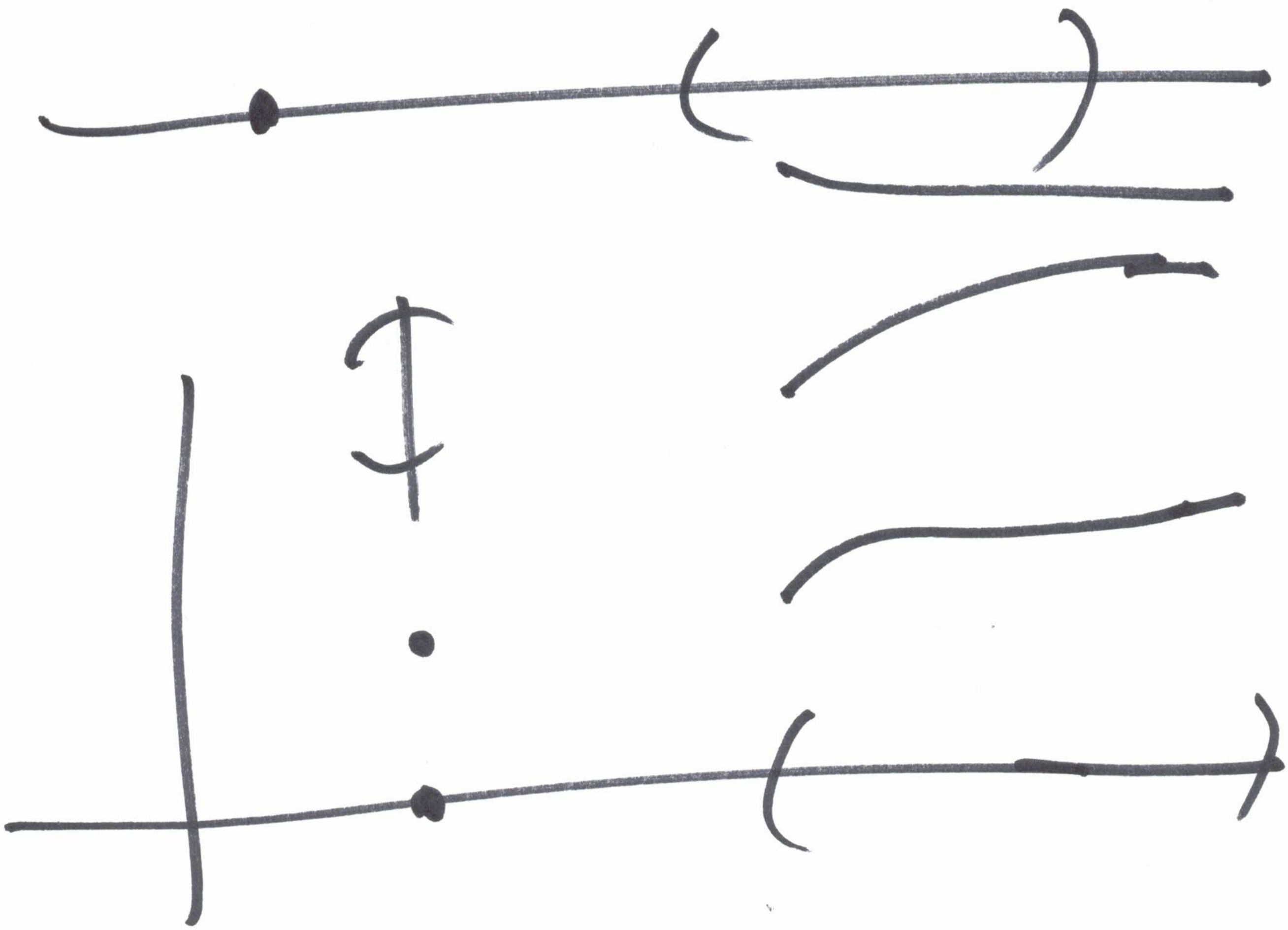
<u>Properties</u>:

Uniform finiteness.

<u>Definable</u> <u>family</u>

$X \subset M^k \times M^n$

$X_{\vec{y}} = \{\vec{x} \in M^n : (\vec{y}, \vec{x}) \in X\}$.



$X = \{(x, f(x)) \in \mathbb{R}^2,\ x \in [0,1]\}$

f analytic.

not algebraic

$X \cap C_d$

Examples

$\mathbb{R}_{alg} = (\mathbb{R}, <, +, \times, 0, 1)$.

$\mathbb{R}_{an} = (\mathbb{R}, <, +, \times, 0, 1$

$\{f : B \to \mathbb{R}\}$

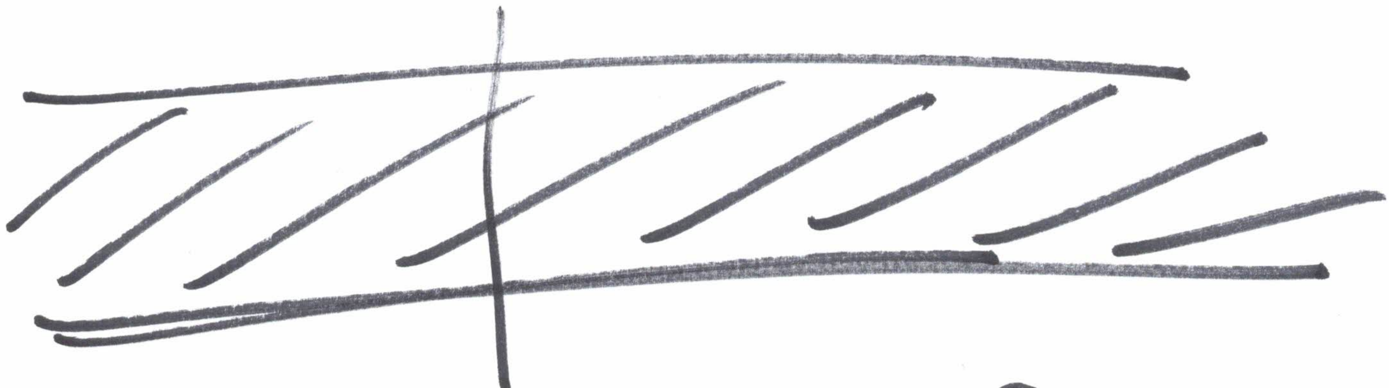
all closed bdd boxes $B \subset \mathbb{R}^n$, all $f : B \to \mathbb{R}$ real analytic on nbd of B.

Gabrielov's Thm.

$\mathbb{R}_{exp} = (\mathbb{R}, <, +, \times, 0, 1, \exp)$

$\underline{\mathbb{R}_{an, exp}} = (\mathbb{R}, <, \ldots f : B \ldots \exp \ldots\}$.

$\exp: \mathbb{C} \to \mathbb{C}$
$\quad\;\; \| $
$\quad\; \mathbb{R}^2$



definable in $\mathbb{R}_{an}$.

Counting Theorem

$Z \subset (0,1)^n$ is definable

$\dim Z = k$

$r \geq 1$,

An r-parametrization of Z is a finite set $\Phi = \{\phi\}$

$\phi : (0,1)^k \to (0,1)^n$

$Z = \bigcup \phi((0,1)^k)$

and ϕ are differentiable to order r and all partial derivs of order $\leq r$ are bdd in abs value by 1.

<u>Definition</u>:

Let $Z \subset P \times (0,1)^n$ family, a definable r-parametrization of Z is a finite set $\Phi = \{\phi\}$ of maps $\phi : P \times (0,1)^k \to (0,1)^n$ such that for each $\vec{y} \in P$, $\phi|_{\vec{y}} : \{\vec{y}\} \times (0,1)^k \to (0,1)^n$ is an r-parametrization of $Z_{\vec{y}}$.

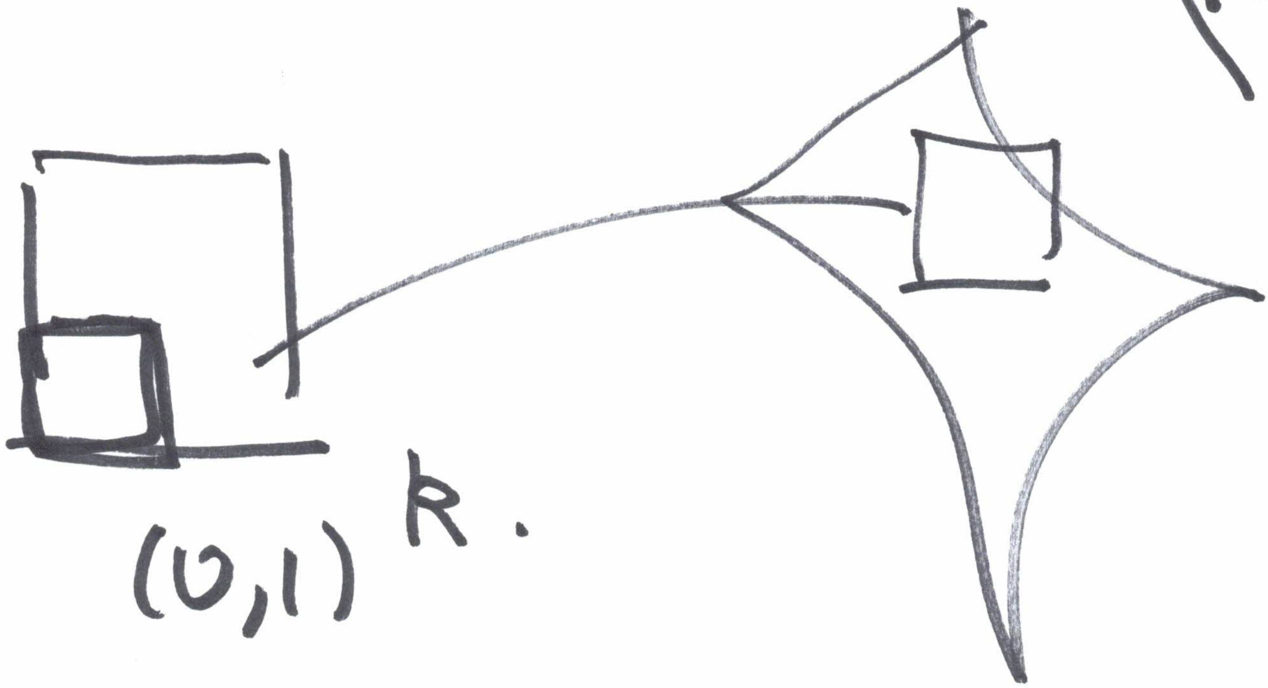
Theorem:
Let Z is a definable family of sets in $(0,1)^n$ $r \geq 1$, then there exists a definable r-parametrization

Yomdin Gromov.

$$Z \subset P \times \mathbb{R}^n$$

$$\rightsquigarrow Z \subset P \times (0,1)^n.$$

$$(0,1)^k \to (0,1)^n.$$



$Z_{\vec{y}} \cap W$

Theorem: $Z \subset P \times \mathbb{R}^n$ definable family $\varepsilon > 0$,

$N(Z_{\vec{y}}^{\text{tran}}, H) \leq c(Z, k, \varepsilon) H^{\varepsilon}$.

$k = \max [\mathbb{Q}(y_i) : \mathbb{Q}]$.