ZP for <u>curve</u>

$$V \subset \mathbb{G}_m^n$$

$$= \{(t,\ 1+t^2,\ 1-t,\ 1+t+t^3,\ 2,\ 3,\ 5,\ 7) \in \mathbb{G}_m^8 : t \in \mathbb{C}\}$$

....?

$V \not\subseteq$ any proper alg sbgp.

ZP says: V has only finitely many intersections with subgroups of codim ≥ 2

i.e. V has only finitely many points that satisfy 2 independent multiplicative conditions.

Examples a lá <u>BMZ</u>

<u>Theorem</u> (Maurin).
ZP ~~X~~ holds for curve $V/\overline{\mathbb{Q}} \subset \mathbb{G}_m^n$.

BMZ: $V/\mathbb{C}$.

For $V/\mathbb{C}$

$2, \pi, t, t-1, t-\pi$

ZP for curve $V_{\mathbb{C}}(\Gamma(1))^\wedge$

is open!

Various special cases known.

$\Gamma(1) = \mathbb{C}$

$Y(1)^n$ special subvarieties.

$Y(1)$ parameterizes elliptic curves$/\mathbb{C}$.

$E = \Lambda \backslash \mathbb{C}$

$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

scale $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$,

$\tau \in \mathbb{H}$

$E: \quad y^2 = x^3 + Ax + B$

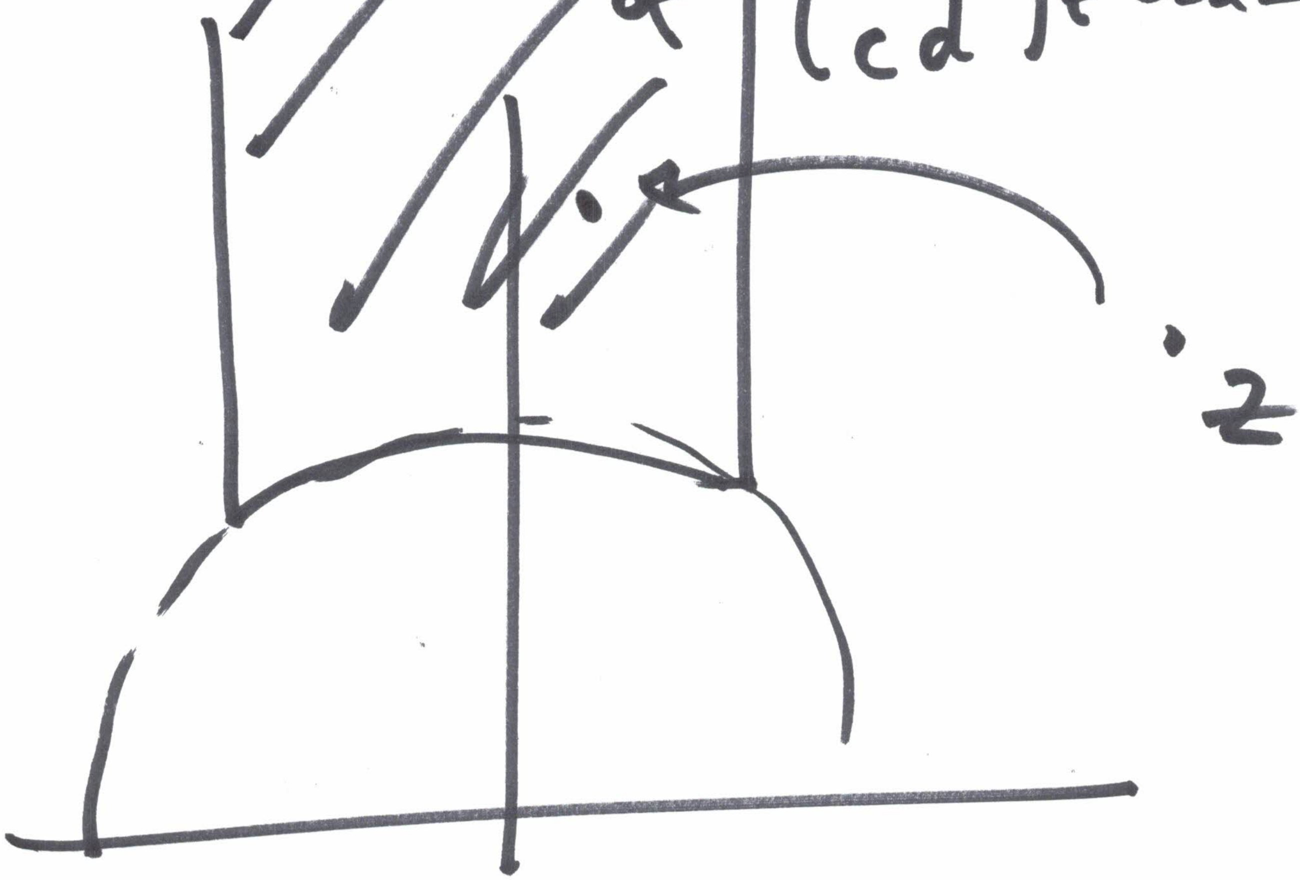
j invariant: $\underline{\underline{j(\tau)}}$.

$j : \mathbb{H} \to \mathbb{C}$

invariant under action of $SL_2(\mathbb{Z})$.

$$j\left(\frac{az+b}{cz+d}\right) = j(z)$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$



If E elliptic curve, endomorphisms of E correspond to λ: $\lambda\Lambda \subseteq \Lambda$

in general, only have $\lambda \in \mathbb{Z}$

Sometimes, other λ: This happens iff $\mathbb{H}$ ∋ $\mathbb{Z} + \mathbb{Z}\tau$ has quadratic τ

$[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$.

Then $j(\tau)$ is "special" is a singular modulus the <u>singular moduli</u>, Σ_1, are the analogues of torsion points in $\mathbb{G}_m$.

If E, E' are elliptic curves, might have $\phi: E \to E'$ with cyclic kernel degree N: controlled by

$$\Phi_N(j(E), j(E')) = 0.$$

For each $N \geq 1$,
there is such <u>modular</u>
polynomial $\Phi_N(X,Y)$.

$\overline{N \geq 2}$, Φ_N is <u>symmetric</u> $\in \underline{\underline{\mathbb{Z}[X,Y]}}$

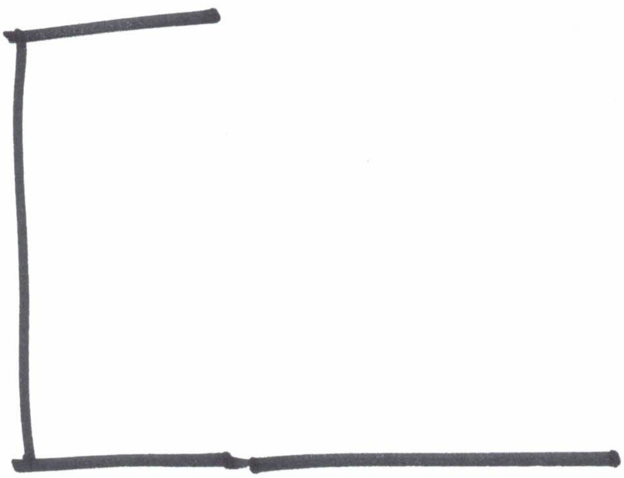
In $Y(1)^2 = \mathbb{C}^2$,
special subvarieties
are:
- Sp pts Σ^2
- Modular curves $\Phi_N(X,Y)=0$
- $\Sigma \times Y(1)$, $Y(1) \times \sigma$, $\sigma \in \Sigma$.
- $Y(1)^2$

Analogue of Lang's Problem for $\mathbb{G}_m^2$

<u>**Theorem**</u> (André) [AO for $Y(1)^2$].

If $V \subset Y(1)^2$ is a curve with infinitely many <u>special</u> <u>points</u> [$V \cap \Sigma^2$] then V is special.

Special subvarieties in $Y(1)^3$



<u>Curve</u> $V \subset Y(1)^3$

A one-dim sp subv. looks like:

- $\Phi_N(X,Y)=0 \quad \Phi_M(Y,Z)=0$
- $X=\sigma\in\Sigma_1, \quad \Phi_N(Y,Z)=0$

$\vdots$

ZP for curve $V \subset Y(1)^3$:

Only finitely many pts on V satisfy two independent modular conditions. (Unless V satisfies such a condition identically.)

If $g \in GL_2(\mathbb{Q})^+$ $(g\mathbb{H} = \mathbb{H})$

$N = N(g)$

~~[illegible]~~ $\Phi_N(j(z), j(gz)) = 0$.

$$\begin{tikzcd} \mathbb{H}^n \arrow[d, "j"'] \\ Y(1)^n \end{tikzcd}$$

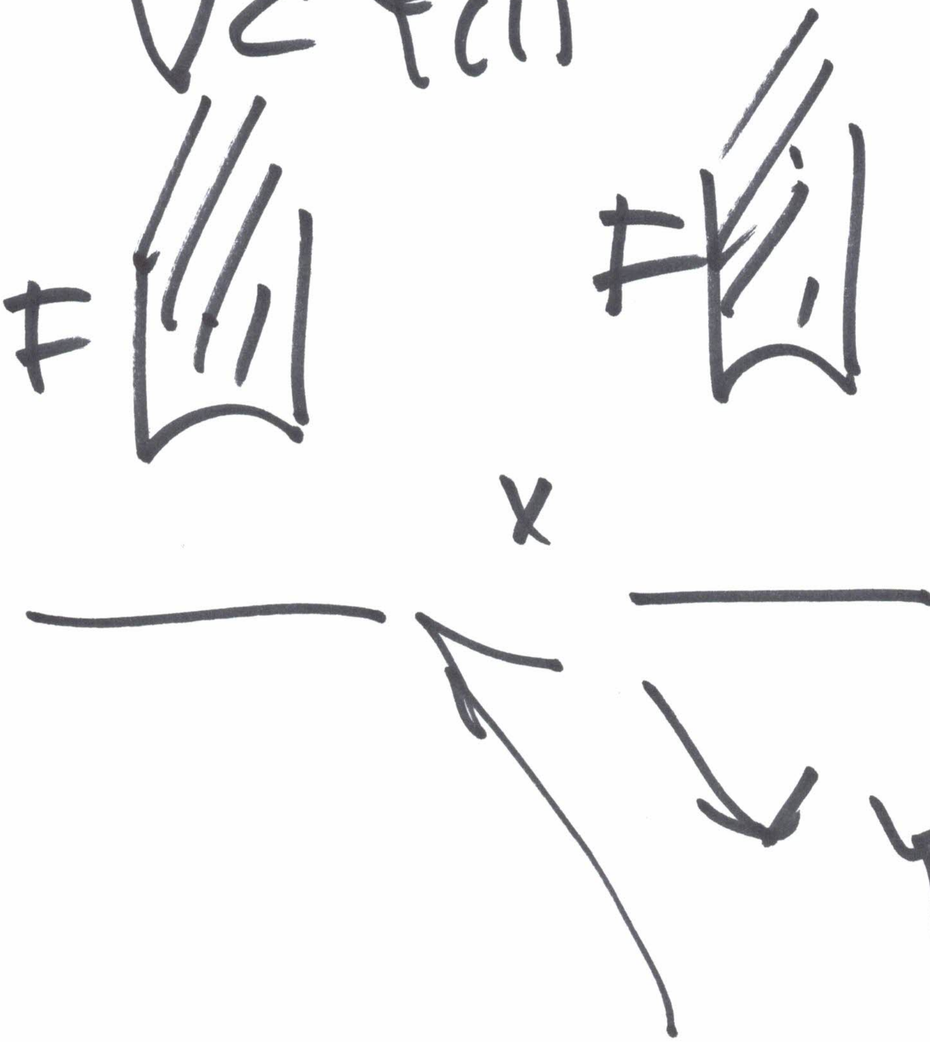
$GL_2(\mathbb{Q})^+$ relations

special

<u>Point</u> <u>counting</u> <u>approach</u>.

$$V \subset Y(1)^3$$

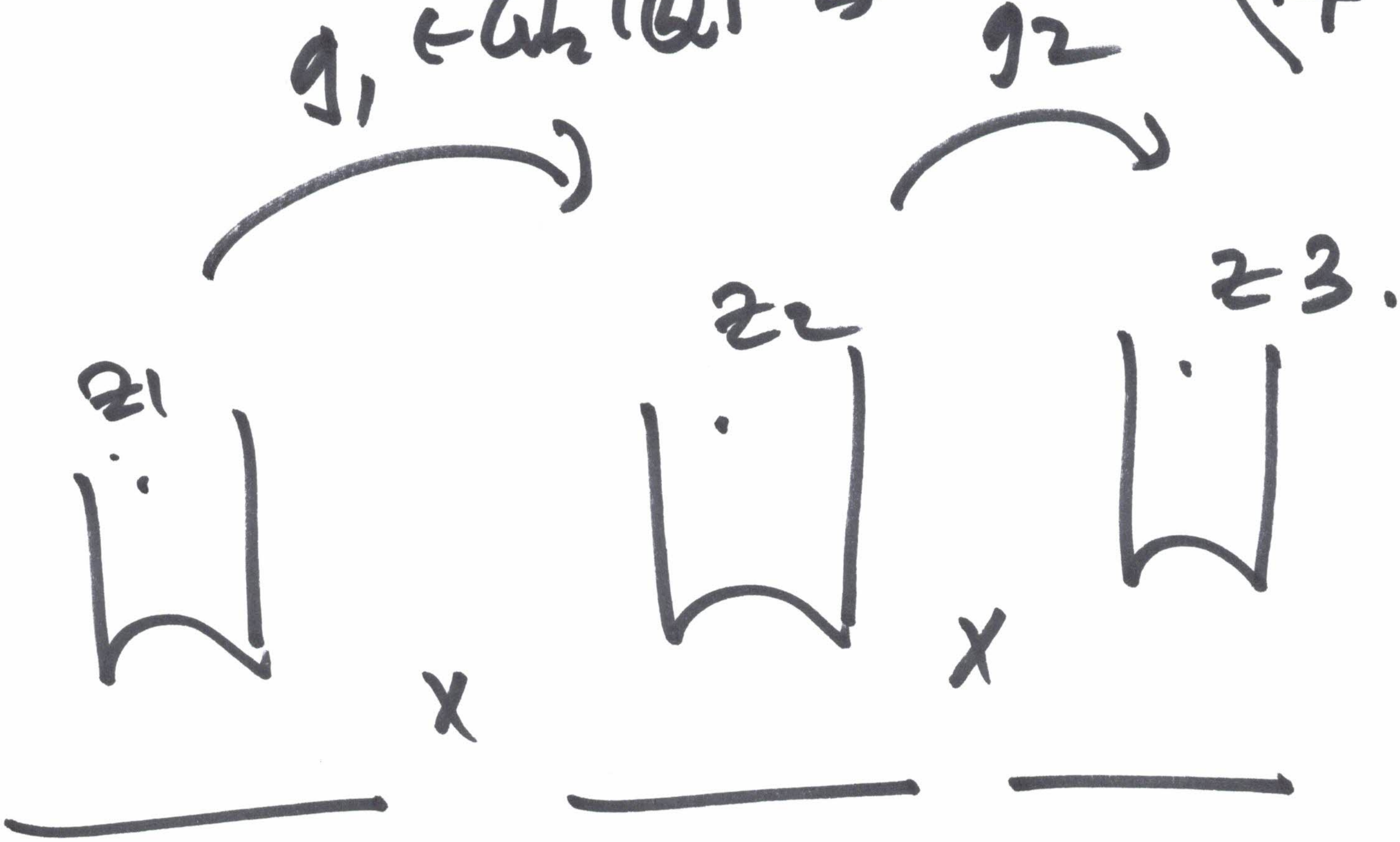
$V \subset Y(1)^2.$



$Y(1)^2.$

$(x_1, x_2) \in V$

$\gamma'(V) \cap \mathcal{F}^2$



$$Y(1)^3 \supset V \to (x_1, x_2, x_3)$$

unlikely $\underbrace{\Phi_N(x_1, x_2)}_{=0}$, $\underbrace{\Phi_N(x_2, x_3)}_{=0}$.

So (x_1, x_2, x_3) "unlikely"

leads to $\Gamma_1, \Gamma_2 < GL_2(\mathbb{Q})^+$

...

$Z = \gamma^{-1}(V) \cap F^3$

For $(\alpha, \beta) \in (\cancel{GL_2(\mathbb{Q})^+})^2$

$Y_{\alpha,\beta} = \{(z_1, z_2, z_3) \in \mathbb{H}^3 :$

$z_2 = \alpha z_1, \; z_3 = \beta z_2$

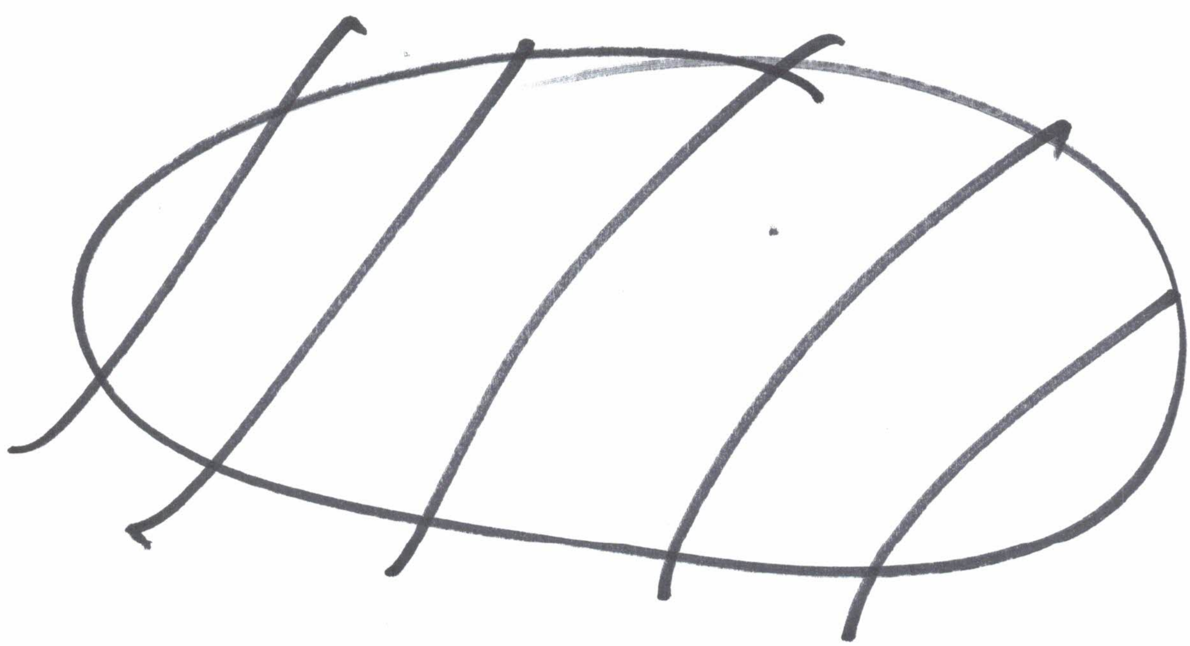
$W = \{(\alpha, \beta) : \cancel{GL_2(\mathbb{Q})^+} \; (GL_2(\mathbb{R})^+)^2 :$

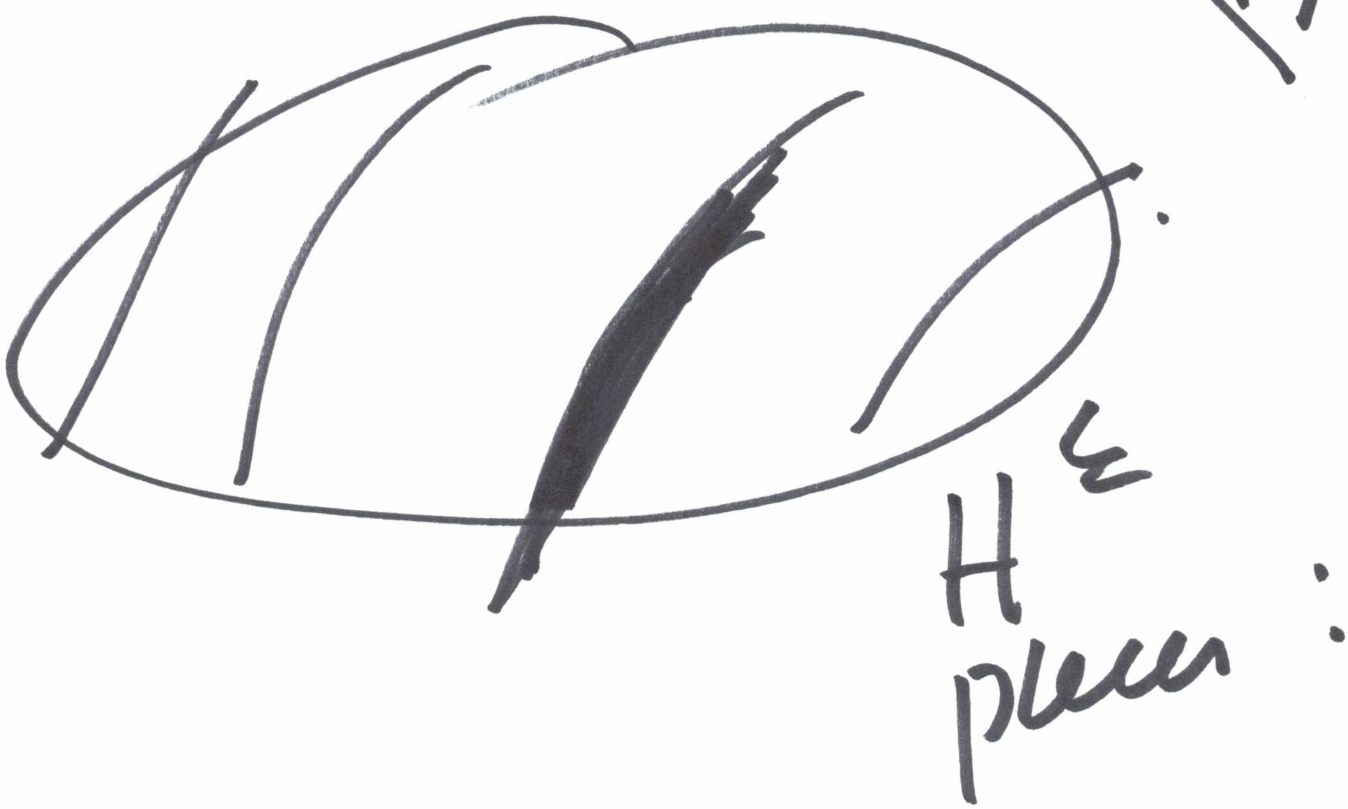
$Y_{\alpha,\beta} \cap Z \neq \emptyset\}.$

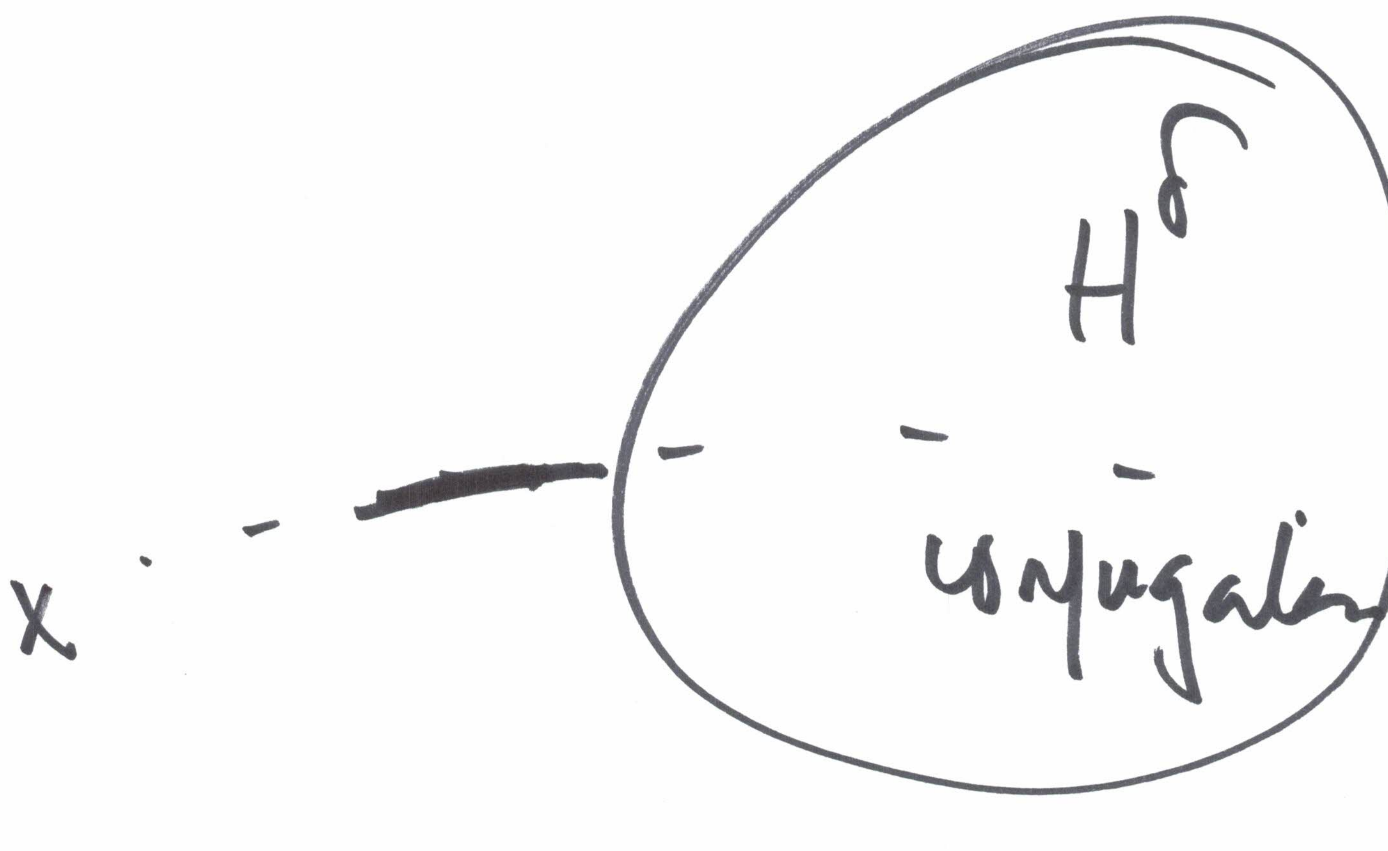
A problem ..

$$W^{alg} = W$$

Counting theorem says something more precise...




H^ε


H^δ
x