1

$$w^5 + x^5 = y^5 + z^5$$

all solutions are "trivial": $\{w, x\} = \{y, z\}$ believed.

$$H\left(q\right) = \max\left(|a|, |b|\right)$$

$q = a/b$ lowest term

Theorem: For $\varepsilon > 0$, $H \geq 1$ there are $\ll_\varepsilon H^{13/8 + \varepsilon}$ nontrivial solutions.

Analogue for certain non algebraic sets in $\mathbb{R}^n$.

Curves

Theorem: Let $f(x)$ be a non-algebraic function that is real analytic on $[0,1]$, $X \subset \mathbb{R}^2$ graph, $\varepsilon > 0$. Then exists a constant $c(f,\varepsilon)$:

$$N(X,H) \leq c(f,\varepsilon) H^{\varepsilon}$$

$$\| $$

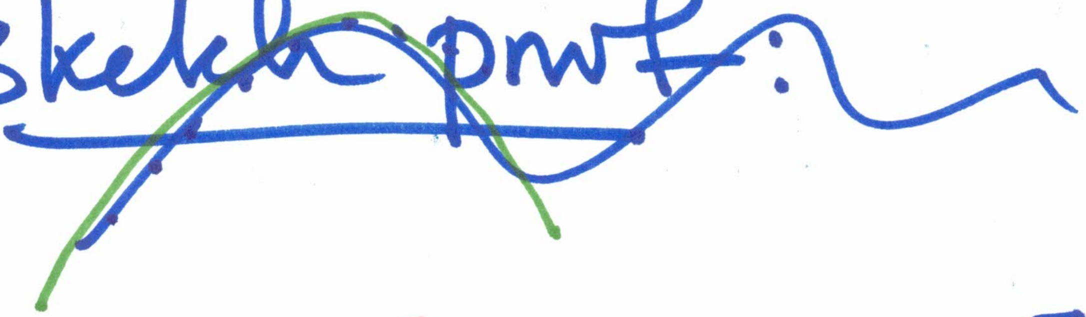
$$\#\{\bar{x} \in X \cap \mathbb{Q}^n : H(\bar{x}) \leq H\}.$$

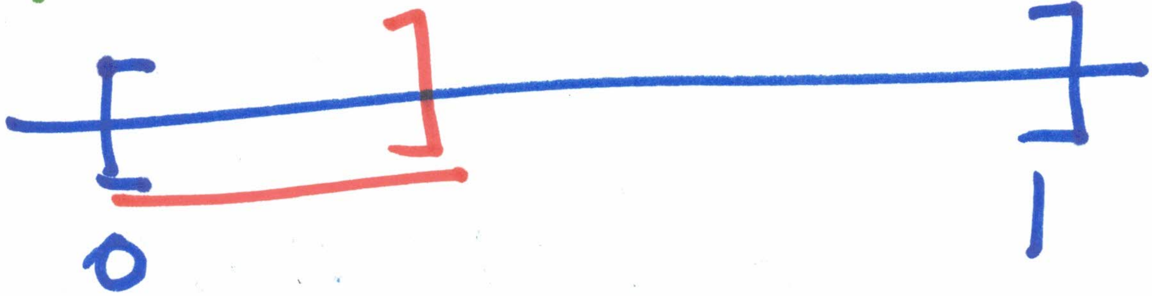
$$\cap$$

$$\mathbb{R}^n$$

"few"

sketch proof:





$[0, L]$

H^{ε} subintervals cover $[0,1]$

Given ε. Choose d $\left(\frac{8}{d+3} < \varepsilon\right)$

$$D = \frac{(d+1)(d+2)}{2}$$

$\#$ monomials in x, y degree $\le d$

$x^a f(x)^b$ $\quad 0 \le a, b \le a+b \le d$.

D

$[0, L]$

all pts of $X \cap \mathbb{Q}^n$

up to height H

suppose D of them

$x_1, \ldots, x_D$

$$\Delta = \det\begin{pmatrix} \ldots x_i^a f(x_i)^b \ldots \end{pmatrix}$$

$D \times D$

$$\Delta = \det\left(\phi_j(x_i)\right)$$

$$\phi_1, \ldots, \phi_D \ , \quad x_1, \ldots, x_D$$

<u>H A Schwarz</u>

$$\Delta = V(x_1, \ldots, x_D) \det\left(\frac{\phi_j^{(i-1)}(\xi_{ij})}{(i-1)!}\right)$$

$$L^{\frac{D(D-1)}{2}}$$

<u>whereas:</u>

row $\quad 1 \quad x_i \quad f(x_i) \quad \ldots \quad f(x_i)^d$

cleared by $\leq (H^{2d})^D$

6

If $|\Delta| \leq \frac{1}{H^{2dD}}$

then $\Delta = 0$.

If $L \asymp H^{-8/d+3}$

In each $[0, L]$.

$\# X \cap C_d \ll c(f, d)$.

$X \subseteq \mathbb{R}^n$.

$f : [0,1]^k \to \mathbb{R}^n$

analytic $\quad X = \operatorname{Im} f$.

Definition

A semi-algebraic set in $\mathbb{R}^n$ is a finite union of sets each defined by finitely many eqns and inegns with real coeffts.

<u>Definition:</u>

For $X \subseteq \mathbb{R}^n$, algebraic part X^{alg} to be union $\bigcup A$ of all connected positive dim semi-alg $A \subseteq X$.

transcendental part

$X^{trans} = X - X^{alg}$.

<u>Theorem:</u> Let $X \subseteq \mathbb{R}^n$ be definable, $\varepsilon > 0$, then exists $c(X, \varepsilon)$:

$$N(X^{trans}, H) \leq c(X, \varepsilon) H^{\varepsilon}$$

Dioph. appl.

F Laurent polynomial in two variables

Lang. $\mathbb{C}[X, X^{-1}, Y, Y^{-1}]$

$$V = \{(x,y) : (\mathbb{C}^\times)^2 : F(x,y) = 0\}.$$

Points of V that are torsion pts in $(\mathbb{C}^\times)^2$

i.e. (ξ, η) roots of unity

Theorem (Ihara, Serre, Tate)

The number of such points is finite unless F is form $x^n y^m = \zeta$, $n, m \in \mathbb{Z}$ not both zero, ζ root of unity.

Such sets $x^n y^m = \zeta$ "torsion coset"

(Laurent, IP3)

Theorem

Let $V \subset X = (\mathbb{C}^\times)^n$

~~X_{tors}~~ torsion points.

Alg subgps

$x_1^{k_1} \ldots x_n^{k_n} = 1$

torsion cosets $= \zeta$

There are finitely many torsion cosets $X_i \subseteq V$ which account for all torsion pts of X in V.

The point-counting approach:
(strategy: Zannier)

$e : \mathbb{C}^n \to (\mathbb{C}^\times)^n$

$e(z_1, \dots, z_n) = (e^{2\pi i z_1}, \dots, e^{2\pi i z_n})$

Studying torsion pts on V
= studying rational pts
on $e^{-1}(V)$

$F = \{(z_1, \dots, z_n) \in \mathbb{C}^n :$
$0 \le \operatorname{Re} z_i < 1\}.$

$e^{-1}(V) \cap F$ is <u>definable</u>

$\mathbb{Z}$ //

$e : \mathbb{Z} \longrightarrow V^*/\overline{\mathbb{Q}}$

if $(\zeta_1, \ldots, \zeta_n) \in V^*$

order $(N_1, \ldots, N_n)$

max : N.

Hardy & Wright

$[\mathbb{Q}(\zeta) : \mathbb{Q}] \geq c(\delta) N^{1-\delta}$

$\mathbb{Z}^{alg}$

if $A \subseteq Z$

$$W \subseteq e^{-1}(V)$$

$$e(W) \subseteq V$$

$$\rotatebox{90}{$\subseteq$}$$

$$\mathbb{C}^n$$

$$\{(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})\} \subset \mathbb{C}^{2n}$$

$$z = x^y$$

$$x, y \in [1, 2]$$