\[ w^5 + x^5 = y^5 + z^5 \]

All solutions are "trivial": \( \{w, x\} = \{y, z\} \) believed.

\[ H(q) = \max (|a_1|, |b_1|) \]

Theorem: For \( \varepsilon > 0 \), \( H \geq 1 \), there are \( \ll H^{13/8 + \varepsilon} \) nontrivial solutions.
Analogue for certain non-algebraic sets in $\mathbb{R}^n$.

**Cubes**

**Theorem:** Let $f(x)$ be a non-algebraic function that is real analytic on $[0,1]^n$, $x \in \mathbb{R}^n$ graph, $\epsilon > 0$. Then exist a constant $c(f, \epsilon)$. $\exists N(x, H) \leq c(f, \epsilon) H \leq 1$

$\# \{ x \in x \cap \phi^n : H(x) \leq H \}$

"few"
Sketch proof:

$[0, L]$ $H \subseteq$ bounded, cover $[0, 1]$

Given $\epsilon$, choose $d$ such that $\frac{2}{d+3} < \epsilon$

$D = \frac{(d+1)(d+2)}{2}$

# monomials in $xy$ degree at most $d$

$x^a f(x)^b$ with $0 \leq a, b \leq d$

\[ \square \]
If $Q \subseteq \mathbb{R}^n$ is compact and $f : Q \rightarrow \mathbb{R}$ is continuous, then $f$ is uniformly continuous on $Q$. Suppose $D$ is a partition of $Q$ by intervals $x_1, \ldots, x_D$. Define

$$\Delta := \det \left( \begin{array}{c}
\ldots 
\vdots 
\end{array} 
\begin{array}{c}
x_1 
\vdots 
\end{array} 
\begin{array}{c}
f(x_1) 
\vdots 
\end{array} 
\ldots 
\right)_{D \times D}$$
\[ \Delta = \det (\phi_i (x_i)) \]
\[ \phi_1, \ldots, \phi_D, \ x_1, \ldots, x_D \]

H & X Schwarz
\[ \Delta = V(x_1, \ldots, x_D) \det (\phi_i(x_i) \phi_j(x_j)) \]
\[ \frac{D(D-1)}{2} \]

Whereas:
\[ \text{row } i \times f(x_i) \ldots \]
\[ \text{cleared by } \leq (H^2d)^D \]
\[ |\Delta| \leq \frac{1}{H^{2dD}} \]

then \( \Delta = 0 \)

\[ H \leq H^{-\frac{8}{d+3}} \leq \quad \quad \]

\[ L = H \]

In each \([0, L]\),

\[ \# X \cap C_p \leq c(f, d). \]
\[ x \leq \mathbb{R}^n \]

\[ f: \mathbb{R}^k \rightarrow \mathbb{R}^n \]

Analytic

\[ X = \lim f \]

**Definition**

A semi-algebraic set in \( \mathbb{R}^n \) is a finite union of sets each defined by finitely many equations and inequalities with real coefficients.
Definition: For $X \subseteq \mathbb{R}^n$, algebraic part $X_{\text{alg}}$ to be union $U_A$ of all connected positive dim semi-alg $A \subseteq X$.

Transcendental part $X_{\text{trans}} = X - X_{\text{alg}}$.

Theorem: Let $X \subseteq \mathbb{R}^n$ be definable, $\varepsilon > 0$, then exists $c(X, \varepsilon)$: $\exists N(X_{\text{trans}}, H) \leq c(X, \varepsilon) H$. 

Dioph. appl.

F Laurent polynomial
in two variables
\[ C[X, X', Y, Y'] \]

\[ V = \{(x, y) : (C_x)^2 : \phi(x, y) = 0 \} \]

Points in \( V \) that are
torsor pts in \( (C_x)^2 \)
i.e. \((s, n)\) rank \( r \) unit
Theorem (Ihara, Serre-Tate)

The number of such pants is finite unless 

\[ F \text{ is Bm } x^m y^m = s \]

n, m \in \mathbb{Z} \text{ not both zero, }

s \text{ not of unity.}

\[ \text{Such sets } x^n y^m = 1 \]

\text{union coset}
Theorem (Laurent, 1831)

Let $V \subset X = (C^\times)^n$

\[ x \text{-tes tangent points.} \]

Alg subgps

\[ x_1, k_1 \ldots x_n, k_n = 1 \]

torus cosets

\[ = \frac{1}{2} \]

There are finitely many torsion cosets $x_i \in V$ which account for all torsion pts of $X$ in $V$. 
The point-counting approach

(Strategy: Zimmer)

\[ e : C^n \to (C^\times)^n \]

\[ e(\mathbf{z}_1, \ldots, \mathbf{z}_n) = (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n}) \]

Studying torsion pts on \( V \)

= studying rational pt on \( e^{-1}(V) \)

\[ F = \{ (z_1, \ldots, z_n) \in C^n : 0 \leq \text{Re} z_i < 1 \} \]

\( e^{-1}(V) \cap F \) is definable

\( \mathbb{Z} \)
\[ e : \mathbb{Z} \to \sqrt{\mathbb{Q}} \]

Let \((S_1, \ldots, S_n) \in \mathbb{V}^*\)

order \((N_1, \ldots, N_n)\)

\[ \max : N. \]

Hardy & Wright

\[ [\Omega(L) : \Omega] \geq \delta \, N^{\gamma} \]

2alg
\[ A \leq z \leq \sqrt[4]{W} \leq e^{-1}(V) \leq v \leq \frac{1}{\sqrt[4]{m}} \]
\{ (x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \}

\subset C_{2n}

\overline{z = x^Y}

x, y \in \mathbb{C}/\{1, 27\}