ARIZONA WINTER SCHOOL PROBLEM SESSION ON MODEL THEORY

ADELE PADGETT

1. Day 1: Basic model theory of $\mathbb{C}$ and $\mathbb{R}$

1.1. Basic model theory.

(1) Let $\varphi(x)$ be an $\mathcal{L}$-formula and n a natural number. Show that there is an $\mathcal{L}$-sentence ψ such that $M \vDash \psi$ if and only if the definable set $Y = \{a \in M : M \vDash \varphi(a)\}$ has at least n elements. What about expressing that Y has at most n elements? Exactly n elements? Could you write a sentence saying Y has infinitely many elements?

(2) Let $A \subset M^{m+n}$ be a definable set and fix $x \in M^m$. Show that $A_x = \{y \in M^n : (x, y) \in A\}$ is definable.

(3) Let $F : M^{m+n} \to M^k$ be a definable function. Show that the set $\{a \in M^m : F(a, \cdot) : M^n \to M^k \text{ is injective}\}$ is definable. Instead of injectivity, what are some other definable properties that we could require of $F(a, \cdot)$? Can you think of properties that would not be definable?

(4) Let $\mathcal{L} = \{+, -, \cdot, 0, 1\}$, and let T be the theory of fields. Let $\varphi(x) := \exists y(xy = 1)$. Find a quantifier-free formula $\psi(x)$ such that $T \vDash \forall x(\varphi(x) \longleftrightarrow \psi(x))$.

(5) Let $\mathcal{L} = \{+, -, \cdot, 0, 1\}$. Let $\varphi(x) := \exists y(x = y^2)$.
   (a) Find a quantifier-free $\mathcal{L}$-formula that is equivalent to $\varphi(x)$ in $\mathrm{Th}(\mathbb{C})$.
   (b) Let $\mathcal{L}_{or} = \mathcal{L} \cup \{<\}$. Find a quantifier-free $\mathcal{L}_{or}$-formula that is equivalent to $\varphi(x)$ in $\mathrm{Th}(\mathbb{R})$.
   (c) Is it possible to find a quantifier-free $\mathcal{L}_{or}$-formula equivalent to $\varphi(x)$ in $\mathrm{Th}(\mathbb{Q})$?

(6) Let T be a $\mathcal{L}$-theory with quantifier elimination. Let $\mathcal{L}_c = \mathcal{L} \cup \{c\}$, where c is a new constant symbol. Let T_c be any $\mathcal{L}_c$-theory that extends T. Show that T_c has quantifier elimination.

1.2. ACF_0.

(1) Find quantifier-free equivalents to the following in ACF_0:
   (a) $\exists x(ax^2 + bx + c = 0)$
   (b) $\exists y(y^3(s^2 + t(t-1)(t-2)) + t^2 = 0)$

(2) Let $\mathcal{L} = \{+, \cdot, 0, 1\}$ be the language of rings, consider $\mathbb{C}$ as an $\mathcal{L}$-structure, and let T be its complete $\mathcal{L}$-theory. Let

$$\varphi(a, b, c) := \exists x_1 \exists x_2 \exists x_3 \left(\bigwedge_{i=1}^{3} x_i^3 + a x_i^2 + b x_i + c = 0 \wedge \bigwedge_{i<j} x_i \neq x_j \right).$$

Find a quantifier-free formula $\psi(a, b, c)$ such that

$$T \vDash \forall a \forall b \forall c(\varphi(a, b, c) \longleftrightarrow \psi(a, b, c)).$$

(3) Let $K \vDash \mathrm{ACF}_0$. Show that every definable subset of K is either finite or cofinite.

Date: February 24, 2023.

(4) Let $\mathcal{L}_c = \{+, \cdot, 0, 1, c\}$ extend the language of rings by a new constant symbol c. Let $\mathrm{ACF}_0(c)$ be the theory ACF_0, but viewed as an $\mathcal{L}_c$-theory (i.e., no axioms are added that say anything about c). Then $\mathrm{ACF}_0(c)$ is not complete (why?). Describe all the completions of $\mathrm{ACF}_0(c)$. Hint: Use exercise 1.1(6).

(5) Let $\mathcal{L} = \{+, \cdot, 0, 1, \exp\}$ be the language of exponential rings. Show that the set of integers $\mathbb{Z}$ is $\mathcal{L}$-definable in $\mathbb{C}_{\exp}$. (This is exercise 2.1 of Scanlon's notes.)

1.3. RCF.

(1) Consider the language $\mathcal{L} = \{<, +, \cdot, 0, 1, f\}$, where f is a function symbol in one variable. We can think of $\mathbb{R}$ as an $\mathcal{L}$-structure by interpreting f as a function $F : \mathbb{R} \to \mathbb{R}$. Write down $\mathcal{L}$-sentences asserting the following:
   (a) $\lim_{x\to 0} F(x) = 1$
   (b) F is continuous on $\mathbb{R}$.

   Write down $\mathcal{L}$-formulas defining the following sets:
   (c) The set of points where F is continuous.
   (d) The set of points where F is differentiable.
   (e) The set of points where F is k-times differentiable.

   Can you define the smooth points of F?

(2) Let $\mathcal{L} = \{< +, \cdot, 0, 1\}$ and consider $\mathbb{R}$ as an $\mathcal{L}$-structure. Let $A \subset \mathbb{R}^n$ be a definable set. Show that the closure of A (in the Euclidean topology) is a definable set.

(3) Let $\mathcal{L} = \{<, +, \cdot, 0, 1\}$. Find quantifier-free equivalents to the following:
   (a) $\varphi(x) := \exists y(y = x^2 + 1)$
   (b) $\varphi(x, y) := \exists z(x + z^2 = y)$

(4) Let $\mathcal{L} = \{<, +, \cdot, 0, 1\}$. The interval $I = (-\sqrt{2}, \sqrt{2}) \subset \mathbb{R}$ can be defined by the formula $\varphi(x) := -\sqrt{2} < x < \sqrt{2}$. In this case, $-\sqrt{2}$ and $\sqrt{2}$ are called parameters. Write down a different formula defining I using no parameters.

(5) Let $\mathcal{L} = \{<, \dots\}$, let T be an o-minimal $\mathcal{L}$-theory, and let $M \vDash T$. Suppose $\varphi(x)$ is an $\mathcal{L}$-formula with only finitely many realizations in M. Show that for every $m \in M$ realizing $\varphi(x)$, there is another $\mathcal{L}$-formula $\psi(x)$ such that m is the only realization of $\psi(x)$.

2. Days 2-3: Uniformity and Compactness

2.1. Uniformity.

(1) Let $K \vDash \mathrm{RCF}$ and $A \subset K^{m+n}$ be definable. Show that there is a definable function $f : K^m \to K^n$ such that for all $\bar{x} \in K^m$, if there is some $\bar{y} \in K^n$ such that $(\bar{x}, \bar{y}) \in X$, then $(\bar{x}, f(\bar{x})) \in X$. In fact, we may pick f so that if $\bar{a}, \bar{b} \in K^m$ and

$$\{\bar{y} \in K^n : (\bar{a}, \bar{y}) \in X\} = \{\bar{y} \in K^n : (\bar{b}, \bar{y}) \in X\}$$

then $f(\bar{a}) = f(\bar{b})$. (Hint: Use induction on n, and use o-minimality in the base case.)

Such a function f is called a *definable choice function for* X. In fact, every o-minimal expansion of an ordered group has definable choice functions.

(2) Let $\mathcal{L} = \{<, +, \cdot, 0, 1, (f_i)_{i\in I}\}$ where $(f_i)_{i\in I}$ are function symbols. Suppose we interpret these new function symbols on $\mathbb{R}$ in such a way that the $\mathcal{L}$-theory of $\mathbb{R}$ is o-minimal.

(a) Let $g : \mathbb{R} \to \mathbb{R}$ be a definable function and assume that $g^{-1}(x)$ is a finite set for all $x \in \mathbb{R}$. Show that there is a natural number N such that for all $x \in \mathbb{R}$, $g^{-1}(x)$ has at most N elements.
(b) Let $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be a definable function and assume that $g^{-1}(\bar{x})$ is a finite set for all $\bar{x} \in \mathbb{R}^n$. Show that there is a natural number N such that for all $\bar{x} \in \mathbb{R}^n$, $g^{-1}(\bar{x})$ has at most N elements.

(Hint: Use cell decomposition and induction.)

(3) Let $\{f_a(x) : a \in A\}$ be a definable family of definable functions in $\mathbb{R} \models \mathrm{RCF}$. Show that the set $\{a \in A : \forall x(\dim(f_a^{-1}(x)) = 1)\}$ is definable. Would the same be true for a definable family of definable functions in ACF_0? (As shown in Proposition 2.19 of Scanlon's notes, dimension is a definable condition in ACF_0.)
(4) Show that there is no definable family of subvarieties of $\mathbb{C}^n$ that contains all special (in the sense of the j-function) subvarieties of $\mathbb{C}^n$. (Hint: Look at $\mathbb{C}^2$ first, and think about degrees of modular polynomials.)
(5) Let $f(z)$ be a complex analytic function definable in an o-minimal expansion of the real field (i.e., view $f : \mathbb{R}^2 \to \mathbb{R}^2$ as a definable function of its real and imaginary parts). Show that the isolated singularities of f cannot be essential. (Hint: look at the fibers $f^{-1}(z)$.)
(6) Let $f(z)$ be a meromorphic function on $\mathbb{C}$ definable in an o-minimal expansion of the real field. Show that f is a rational function. Thus any meromorphic function on all of $\mathbb{C}$ which is definable in $\mathbb{R}_{\mathrm{an,exp}}$ must already be definable in just the real field.

2.2. **Compactness.** To expand our discuss of compactness, we will use the notions of *types* and *saturation*.

Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. An *n-type* is a collection p of $\mathcal{L}$-formulas in variables $x_1, \ldots, x_n$ such that there is some model $\mathcal{M} \models T$ and some $a_1, \ldots, a_n \in M$ such that $M \models \varphi(a_1, \ldots, a_n)$ for all $\varphi \in p$. An n-type is *complete* if for all $\mathcal{L}$-formulas φ in n variables, either $\varphi \in p$ or $\neg\varphi \in p$. Let $\mathcal{M} \models T$. If $A \subset M$, let $\mathcal{L}_A$ be the language $\mathcal{L}$ with new constant symbols added to represent each element of A. We sometimes call an n-type p of $\mathcal{L}_A$ formulas "a type with parameters from A" or "a type over A". We say that $\mathcal{M}$ is *saturated* if for all $A \subset M$ with $|A| < |M|$, all $n \in \mathbb{N}$, and all complete n-types p with parameters from A, there is some $\bar{a} \in M^n$ such that $M \models \varphi(\bar{a})$ for all $\varphi \in p$.

$\mathbb{C}$ is a saturated model of ACF_0. So for every type over a countable set of parameters, there is some tuple of elements of $\mathbb{C}$ that satisfies every formula in the type.

Example 2.1. This is an example of a typical proof using the compactness theorem, and a non-example of saturation. Let $\mathcal{L} = \{<, +, \cdot, 0, 1\}$, consider $\mathbb{Q}$ as a $\mathcal{L}$-structure, and let T be its $\mathcal{L}$-theory. We will find a type with no parameters and show that no element of $\mathbb{Q}$ satisfies every element of the type. Let p be the following set of formulas in a single variable x:

$$\left\{x > a : \sqrt{2} > a \in \mathbb{Q}\right\} \cup \left\{x < a : \sqrt{2} < a \in \mathbb{Q}\right\}.$$

These formulas seem to use parameters that are not in $\mathcal{L}$, but if, for example, $a = b_1/b_2$ with $b_1, b_2 \in \mathbb{N}$, the formula "$x > a$" is really shorthand for

$$\left(\overbrace{1 + \cdots + 1}^{b_2 \text{ times}}\right) \cdot x > \left(\overbrace{1 + \cdots + 1}^{b_1 \text{ times}}\right)$$

which is an $\mathcal{L}$-formula with no parameters. We will show p is a 1-type and extend it to a complete 1-type. Let c be a new constant symbol that we add to our language. For each finite subset p_0 of p, build an $\mathcal{L}_c$-structure $\mathbb{Q}_c$ by interpreting c to be a rational number close enough to $\sqrt{2}$ so that $\mathbb{Q}_c \vDash \varphi(c)$ for every $\varphi \in p_0$. Then the compactness theorem tells us that there exists an $\mathcal{L}_c$-structure $\mathcal{M} \vDash T \cup \{\varphi(c) : \varphi \in p\}$. Thus p is a 1-type. Let $p' \supset p$ be the set of all unary $\mathcal{L}$-formulas that are true of $c \in \mathcal{M}$. Then p' is a complete 1-type. However, there is no element of $\mathbb{Q}$ that satisfies every formula in p'. Thus $\mathbb{Q}$ is not saturated.

(1) Show that $\mathbb{R}$ is not saturated as a model of RCF.
(2) Show that quasiminimality is a property of structures, not theories, as mentioned in Remark 2.3 of Scanlon's notes.
(3) Let $V \subset \mathbb{C}^m \times \mathbb{C}^n$ be a definable set. For every $q \in \mathbb{C}^n$, we have the fiber over q $V_q \subset \mathbb{C}^m$. Suppose $\{Ti\}_{i\in\mathbb{N}}$ is a countable collection of subvarieties of $\mathbb{C}^m$ such that for each $q \in \mathbb{C}^n$, there exists $k \in \mathbb{N}$ such that $V_q \subset T_1 \cup \cdots \cup T_k$. Show that there exists a uniform bound $k_0 \in \mathbb{N}$ such that for all $q \in \mathbb{C}^n$, $V_q \subset T_1 \cup \cdots \cup T_{k_0}$. (Hint: Use compactness and the fact that $\mathbb{C}$ is saturated.)
(4) We say an $\mathcal{L}$-theory "eliminates $\exists^\infty$" if for any $\mathcal{L}$-formula $\varphi(x, \bar{y})$ there is a natural number n_φ such that for all $\mathcal{M} \vDash T$ and $\bar{a}$ in M, the set $\{x \in M : \varphi(x, \bar{a})\}$ is infinite iff it has more than n_φ elements. Show that ACF_0 and RCF eliminate $\exists^\infty$. (Hint for ACF_0: Suppose not. Then there is some formula $\varphi(x, \bar{y})$ that witnesses this. Use compactness to show that there is $K \vDash \mathrm{ACF}_0$ and $\bar{a}$ in K such that $\{x : \varphi(x, \bar{a})\}$ is neither finite nor cofinite. For RCF, use o-minimality.)

3. Day 4: DCF_0 and More Uniformity

3.1. DCF_0.

(1) Let (K, δ) be a differential field. Show $a_1, \ldots, a_n$ are linearly dependent over the field of constants if and only if their Wronskian vanishes. (We can see the iff statement as a form of quantifier elimination.)
(2) Let R be a differential ring. If $a \in R$ is algebraic over R^δ, then $a \in R^\delta$.
(3) Let $L|K$ be a differential field extension. Suppose $a \in L^\delta$ is algebraic over K. Show a is algebraic over K^δ.
(4) Find a quantifier-free equivalent in DCF_0 to the following formula:
$$\exists x((\delta s = \delta x) \wedge (\delta t = tx + s)).$$

3.2. **Uniform existential closedness.**

Suppose $X \subset \mathbb{C}^n \times \mathbb{R}$ is a definable (uncountable) family of free and broad varieties (as defined toward the end of Section 2 of Scanlon's notes). Suppose each element of the family has points of the form $(b, \exp(b))$.

(1) Show that there is some fundamental domain $\mathcal{F}$ of exp such that the set $\{a \in \mathbb{R} : \exists b \in \mathcal{F} : (b, \exp(b)) \in X_a\}$ is uncountable.
(2) Write a formula $\varphi(y)$ in the language $\mathcal{L} = \{0, 1, +, \cdot, <, \exp\}$ saying "X_y contains a point on the graph of $\exp\restriction_{\mathcal{F}}$".
(3) Show that there is an interval $I \subset \mathbb{R}$ of positive length such that $\varphi(a)$ holds for all $a \in I$.
(4) Using definable choice, show that there is a piecewise continuous function $g : I \to X$ that sends each parameter $a \in I$ to a point of the form $(\bar{b}, \exp(\bar{b}), a) \in X$.

3.3. **Uniform Ax-Schanuel.** Recall Ax-Schanuel for exp (Theorem 4.1 in Scanlon's notes). Show the following uniform version of Ax-Schanuel: Let (K, δ) be a differential field of characteristic 0. Let $(V_b) \subset K^n \times (K^*)^n$ be a definable family. Then there are finitely many special $T_1, \ldots, T_m \subset (K^*)^n$ such that for all $b \in (K^*)^n$ and for all $(x, y) \in V_b$, if

(1) $\delta y_i = y_i \delta x_i$
(2) $\operatorname{tr.deg}_C C(x, y) < n + \operatorname{rk}(\delta_i x_j)$

then $(x, y) \in T_1 \cup \cdots \cup T_m$. (Hint: compactness).