Goal: complex methods from dynamical systems

Intersections - both unlikely & likely

- currents, plurisubharmonic functions (height functions)
- equidistribution results.
Example 0

\[ f: \mathbb{P}^n \to \mathbb{P}^n / \mathbb{C} \]

\[ f = (f_0 : f_1 : \ldots : f_n) \]

\[ \left\{ f_0 = \ldots = f_n = 0 \right\} \quad \text{homog. poly} \]

\[ = \emptyset \quad \text{deg } d \]

Assume \( d > 1 \).

Study \( f^n = f \circ \ldots \circ f \) as \( n \to \infty \)

Orbit of \( z_0 \in \mathbb{P}^n \) \( n \) times

\[ = \{ f^n(z_0) \}_{n \geq 0} \]
Example 0

\[ N = 1 \quad f(z) = z^2 \quad f: \mathbb{P}^1 \to \mathbb{P}^1 \]

\[ l_z < 1 \Rightarrow f^n(z) \to 0 \]

\[ l_z > 1 \Rightarrow f^n(z) \to \infty \]

\[ \text{In } \mathbb{C}^b, a \text{ point } z_0 \text{ has finite orbit } \]

\[ \Leftrightarrow z_0 \text{ is a root of unity.} \]

\[ f|_{S^1} \text{ is chaotic} \]

\[ f(e^{2\pi i \theta}) = e^{2\pi i (2\theta)} \]
Example 1 - Lattès maps

$E = \text{elliptic curve } / \mathbb{C}$

eg. $E_t = \{ y^2 = x(x-1)(x-t) \}$

$t \in \mathbb{C} \setminus \{0,1\}$

Take $\psi : E \to E$

$\psi(P) = P + P = 2P$
\[ \psi(P) = 2P \]  \hspace{1cm} \deg \pi = 2 \]

\[ \pi(P) = \pi(-P) \]  \hspace{1cm} \deg f = 4 \]

Example - \[ E_t \quad \pi(x, y) = x \]

\[ f_t(x) = \frac{(x^2 - \frac{1}{4})^2}{4 \cdot x (x-1)(x-t)} \]

Called Latte's examples
Observation  A point $P \in E$ has finite orbit for $f$ if and only if $P$ is torsion.

\[
\left( P, 2P, 4P, 8P, \ldots \right) = 2^n P = 2^m P
\]

$\iff \pi(P) \in \mathbb{P}^1$ has finite orbit for $f$. 

Preperiodic $= \text{finite orbit}$
Lattès observed (1918)

Preperiodic points for this $f$ are dense

Julia set of $f$, if $\mathbb{P}^1$.

Upward chaotic set.

$f : \mathbb{P}^n \to \mathbb{P}^n$ is Lattès if $\exists$ abelian variety $A$

$A \to A \quad G \subset \text{Aut} A$

$\pi \downarrow f \downarrow \pi$

$\mathbb{P}^n \to \mathbb{P}^n = A / G$
For $N > 1$, these examples are rare.

Remark (Fakhruddin)

$\forall \ A \xrightarrow{\varphi} A \quad \varphi(P) = 2P$

$\exists \ A \leftrightarrow \mathbb{P}^M$

so $\varphi$ extends to a morphism $f: \mathbb{P}^M \hookrightarrow$. 
Canonical measures

Theorem (Lyubich, Mane', Briend- Duval)

Given $f: \mathbb{P}^N \to \mathbb{P}^N / C$

$\exists !$ probability measure $\mu_f$ on $\mathbb{P}^N(C)$ s.t.

- $\mu_f(V) = 0$ for proper subvariety $V$
- $\frac{1}{d^n} f^* \mu_f = \mu_f$

$\mu_f$ is unique measure of maximal entropy
\[ f_\ast \nu_f (S) = \nu_f (f^{-1}(S)) \]

\[ g : P^n(C) \to \mathbb{R} \text{ cont.} \]

\[ \int g \, d(f_\ast \nu) = \int \bigg( \sum_{f(x) = y} g(x) \bigg) \, \nu_f \]

\[ \int \frac{1}{\text{d}^n \nu_f(S)} f \]

\[ f_\ast \nu_f = \nu_f \]
Example 0

\[ f(z) = z^2 \]

\[ S^1 \]

\[ \mu_f = \text{uniform on } S^1 \]

\[ \mu_f \text{ Haar} \]

Example Lattices

\[ E \xrightarrow{\mu_H} \]

\[ P^+ \]

\[ \pi_+ \mu_H = \mu_f \]