Lecture IV

$$(K_S, x_S)$$

$$f \in \mathcal{A}_c(K_S, x_S) \circlearrowleft H^{\text{spherical}}_{K_x}$$

$$x \in |X| \setminus S.$$

Satake isom:

$$\{ H_{K_x} \longrightarrow \overline{\mathbb{Q}}_\ell \}$$

$$\updownarrow$$

$$\{ \text{ss conj. classes in } \hat{G}(\overline{\mathbb{Q}}_\ell) \}$$

$\forall\, x \notin S, \rightsquigarrow \quad \sigma_x \in \hat{G}_{ss}(\overline{\mathbb{Q}}_\ell)/\sim$

Langlands corr $\rightsquigarrow$

$$\exists\, \rho : \pi_1(X \setminus S) \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\ell)$$

s.t. $\quad \rho(Fr_x)^{ss} \sim \sigma_x$

Construct ρ from f.

Geometrize.

(Drinfeld, Laumon, ...)

$G = GL_n$

$T_x \in H_{K_x}$

$$\mathbb{1}_{K_x \begin{pmatrix} t_x & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} K_x}$$

$f : Bun_G(k) \longrightarrow \overline{\mathbb{Q}}_\ell$

$$(T_x f)(\mathcal{E}) = \sum_{\substack{\mathcal{E}' \hookrightarrow \mathcal{E} \\ \text{length } 1 \\ \text{at } x}} f(\mathcal{E}').$$

$\cong \mathbb{P}(\underline{\mathcal{E}_x})$

$$\begin{tikzcd} & Hk^1_x \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ Bun_G & & Bun_G \times (\underline{\underline{X \setminus S}}) \end{tikzcd}$$

$$Hk^1 = \{ \mathcal{E}' \underset{\text{length } 1}{\hookrightarrow} \mathcal{E} \}$$

T_x geometrized into

$\mathcal{F}$: sheaf on Bun_G

$T_1 \mathcal{F} = h_{2!}(h_1^* \mathcal{F})$: sheaf on $Bun_G \times (X \setminus S)$

Eigensheaf:

$$T_x f = \underline{\underline{\lambda_x}} \cdot f, \quad \lambda_x \in \overline{\mathbb{Q}_\ell}$$

$$T_1 \mathcal{F} = \mathcal{F} \boxtimes E$$

on $X \setminus S$

Goal: compute E.

$$\mathrm{Tr}(\mathrm{Fr}_x, E) = \lambda_x \quad \forall\, x \in X \setminus S.$$

<u>Ex</u>. (Kloosterman auto. datum).

Gross. constructed Kl auto. datum.

showed rigidity. (trace formula)

prediction on ρ.

(Kloo. loc system).

Heinloth-Ngô-Y. construct ρ.

$G = PGL_n$

$K_0 = I_0$, $\chi = 1$.

$$\chi_\infty : K_\infty = I_\infty^+ \longrightarrow k \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^\times$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \\ & \ddots & \ddots & a_{n-1,n} \\ & & \ddots & \\ a_{n1} & & & \ddots \end{bmatrix} \longmapsto \left(\sum a_{i,i+1} + \frac{a_{n1}}{\tau} \right) \bmod$$

$I_\infty \hookleftarrow I_\infty$ ⑤

G/N:

$\downarrow$

G/B:

$\mathrm{Bun}_G(K_0, K_\infty)$.

$$\Vert$$

$\{\ \mathcal{V}$: rk n v.b. on $\mathbb{P}^1$

$F^n \supset F^{n-1} \supset \cdots \supset F^1$ full flag ✓

$\mathcal{V}_0$

$0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathcal{V}_\infty$. ✓

$\boxed{e_i \text{ a basis of } F_i/F_{i-1}}$ $\}./\mathrm{Pic}$

$\exists!$ relevant pt on each comp.

of $\mathrm{Bun}_G(K_0, K_\infty)$.

$\downarrow$

$\deg \mathcal{V} \mod n$.

$\mathcal{E}_0$: $\mathcal{O}^n = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}e_n$

at 0 F^* :

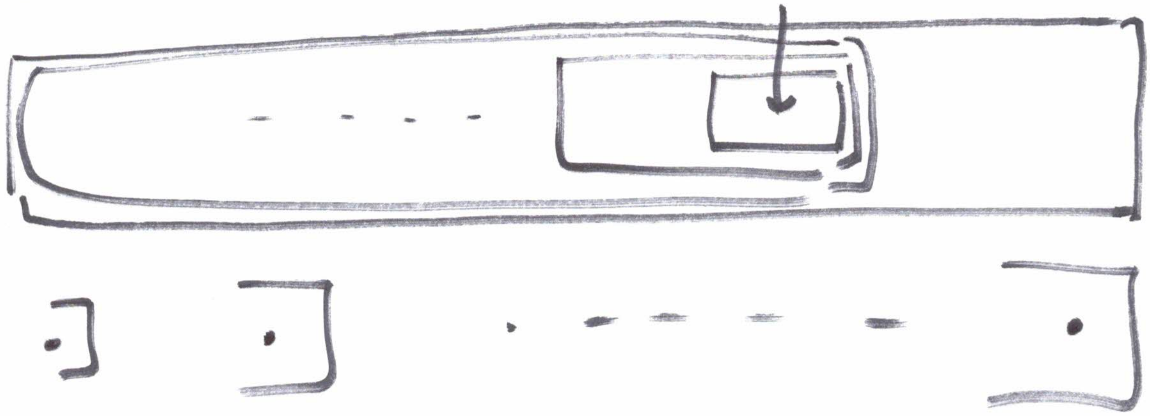
at ∞ F_* :

$\mathrm{Aut}(\mathcal{E}_0) = 1$.

$\mathcal{E}_1 : \quad \mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \cdots \oplus \mathcal{O}e_{n-1} \oplus \mathcal{O}(1)e_n$

F^* :

F_*



$$\mathrm{Hk} \supset \left\{ \mathcal{E}_0 \overset{\varphi}{\hookrightarrow} \mathcal{E}_1 \right\}$$

preserving $F^*, F_*, \ldots$

$$\begin{tikzcd}
 & \mathrm{Hk} \arrow[dl, "ev_\infty"'] \arrow[dr, "\pi"] & \\
\mathbb{A}^n \simeq I^+_\infty / I^{++}_\infty \arrow[d, "\mathrm{sum}"] & & \text{Curve: } \mathbb{P}^1 \setminus \{0,\infty\} \\
\mathbb{A}^1 \quad AS_\psi \arrow[uur, bend left, "(-)^*"] & & E \arrow[from=uul, bend left, "\pi_!"]
\end{tikzcd}$$

$\cancel{K_\infty}$

Supp(coker(φ))

$$\begin{tikzcd} & (\mathbb{G}_m)^n \arrow[dl, "\sigma = \text{sum}"'] \arrow[dr, "\pi = \text{prod.}"] & \\ \mathbb{A}^1 & & \mathbb{G}_m \end{tikzcd}$$

$$E = R^{n-1}\pi_! \, \sigma^* AS_\psi$$

is a rank n local system.

= Deligne's Kloosterman sheaf.

General G.

H_{K_x} has a Kazhdan-Lusztig basis C_λ

$\lambda \in X_*(T)^{dom} \longleftrightarrow$ irred. rep of $\hat{G}$

T_x for $GL_n \rightsquigarrow$ std rep of $\hat{G} = GL_n$

$$\begin{tikzcd} & \mathrm{Hk}_\lambda \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G(K_S) & & \mathrm{Bun}_G(K_S) \times (X \setminus S) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F} \arrow[rr, bend left, "\otimes\, \mathrm{IC}_\lambda"] & & T_\lambda \mathcal{F} \end{tikzcd}$$

$\mathcal{F}$ eigensheaf

$$T_\lambda \mathcal{F} = \mathcal{F} \boxtimes E_\lambda$$

$X \setminus S$.

Fact:

$$\lambda \longmapsto E_\lambda$$

comes from $\hat{G}$-loc sys on $X \setminus S$.

$$\begin{tikzcd} \rho : \pi_1(X \setminus S) \arrow[r] \arrow[dr, "E_\lambda"'] & \hat{G} \arrow[d] \\ & GL(V_\lambda) \end{tikzcd}$$

Applications

$(\underline{K_S}, \underline{\chi_S})$ "tame" auto. datum

(K_x parahoric)

makes sense over any $\underline{k}$.

can construct $\hat{G}$-loc systems

on $\underline{\mathbb{P}^1_k \setminus S}$.

$\rightsquigarrow$ E_8-loc. sys on $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$.

<u>motivic</u>.

$\rightsquigarrow$ inverse Galois problem

$$\mathrm{Gal}(K/\mathbb{Q}) \cong E_8(\mathbb{F}_\ell)$$

$\ell \gg 0$.

"rigidity method".

Open problems:

- Classification of rigid auto. data.
  $\updownarrow$
  rigid loc. sys

  $G = GL_n$.

  Katz, Arinkin

- Checking rigidity.

- weakly rigidity case.
  $\dim \mathcal{A}_c(K_S, \chi_S) > 1$.

$$\left\{ \mathcal{E}'' \overset{\lambda}{\dashrightarrow} \mathcal{E}' \overset{\mu}{\dashrightarrow} \mathcal{E} \right\} \longrightarrow \left\{ \mathcal{E}'' \dashrightarrow \mathcal{E} \right\}$$

$$T_\lambda * T_\mu = \sum C_{\lambda\mu}^{\nu} T_\nu$$