Rigidity

Automorphic Data

- $S \subset |X|$ finite
  - all $k = \mathbb{F}_q$-points.

  (e.g $X = \mathbb{P}^1$, $S = \{0, \infty\}$
  $S = \{0, 1, \infty\}$).

- $x \in S \rightsquigarrow K_x \subset G(F_x)$
  cpt open

- $x \in S$.

$$\begin{tikzcd} K_x \arrow[rr, "\chi_x"] \arrow[dr, two heads] & & \mathbb{C}^* \\ & L_x = \text{finite} \arrow[ur] & \end{tikzcd}$$

(K_S, χ_S).

Given (K_S, χ_S)

(K_S, χ_S) - typical auto forms

If $\chi_S = 1$, these are $f \in \mathcal{A}_{K,c}$

$$K = K_S \times \prod_{x \notin S} G(\mathcal{O}_x)$$

In general.

$$f : G(F) \backslash G(\mathbb{A}) / \prod_{x \notin S} G(\mathcal{O}_x) \xrightarrow{\text{cpt supp}} \mathbb{C}$$

is (K_S, χ_S) - typical, if

$$f(g k_x) = \chi_x(k_x) f(g)$$

$$\forall \ x \in S, \ k_x \in K_x, \ g \in G(\mathbb{A}).$$

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want to make $\dim \mathcal{A}_c(K_S, \chi_S) = 1$.

If $\dim A_c(K_S, \chi_S) = 1$.

$\forall\ y \notin S$.

$$f \in A_c(K_S, \chi_S) \circlearrowleft H_{K_y}$$

f is ~~eigen~~ Hecke eigen.

<u>Ex.</u> $X = \mathbb{P}^1$, $G = SL_2$

$S = \{0, 1, \infty\}$.

$$K_x = I_x \quad \forall\ x \in S.$$

$$I_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{c} a, b, c, d \in \mathcal{O}_x \\ c \in \mathfrak{m}_x \end{array} \right\}$$

$$\begin{array}{ccc} I_x & & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \downarrow & & \downarrow \\ k^\times & \ni & \bar{a} \\ \downarrow{\scriptstyle \chi_x} & & \\ \mathbb{C}^* & & \end{array}$$

> $\chi_0, \chi_1, \chi_\infty$ "generic" position
> $\Rightarrow A_c(K_S, \chi_S)$ has dim 1.

Generic means

$$\chi_0^{\pm 1} \chi_1^{\pm 1} \chi_\infty^{\pm 1} \neq 1$$

$f \in A_c(K_S, \chi_S)$

$\rightsquigarrow$ 2-dim'l local system
on $\mathbb{P}^1 \setminus \{0,1,\infty\}$
"hypergeometric".

$G = \underline{PGl_2}$ $\qquad$ $\chi_0 = \chi_1 = 1$
χ_∞ = quadratic.

$\rightsquigarrow$ Local system

$$\begin{array}{c} \{E_t\} : \ y^2 = x(x-1)(x-t) \\ \downarrow \\ \mathbb{P}^1 \setminus \{0,1,\infty\} \end{array}$$

$\{H^1(E_t)\}$: rk 2 loc. sys. on $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

<u>Ex</u> $X = \mathbb{P}^1$, $S = \{0, \infty\}$, $G = SL_2$

$$K_0 = I_0, \quad \chi_0 = 1.$$

$$K_\infty = I_\infty^+ \qquad \tau = \text{unif. at } \infty$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a, d \equiv 1 \ (m_x) \\ b \in \mathcal{O}_x, \quad c \in m_x \end{array} \right\}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \cancel{k \oplus k}\; k & \ni & b + \frac{c}{\tau} \bmod \tau \\ \downarrow \psi & & \\ \mathbb{C}^* & & \end{array}$$

$$\chi_\infty : K_\infty \longrightarrow k \xrightarrow{\ \psi\ } \mathbb{C}^*$$

$$\Rightarrow \dim A_c(K_S, \chi_S) = 1.$$

$\rightsquigarrow$ Kloosterman loc. sys on $\mathbb{P}^1 \setminus \{0, \infty\}$

$$Kl(a) = \sum_{x \in k^*} \psi\left(x + \frac{a}{x}\right)$$

Naive rigidity of (K_S, χ_S)

$$\dim A_c(K_S, \chi_S) = 1.$$

Issue: • $G \neq$ sc.

Bun$_G$ has several cpts.

• not clear same holds after base change

$k \rightsquigarrow k'$.

Base change of auto. data

k'/k finite. $X' = X \otimes k'$

$S \rightsquigarrow S'$ preimage of S in X'.

$K_x \otimes_k k'$

$$\chi'_x : K_x(\otimes k') \overset{Nm}{\dashrightarrow} K_x \xrightarrow{\chi_x} \mathbb{C}^*$$

Ex 1

$$I_x \longrightarrow k^\times \xrightarrow{\chi} \mathbb{C}^*$$

$$\begin{tikzcd} I_x' = I_x(k') \arrow[r] & k'^\times \arrow[r, "Nm"] & k^\times \arrow[d, "\chi"] \\ & & \mathbb{C}^* \end{tikzcd}$$

~~ent~~

$a, b, c, d \in \cancel{k}\, \mathcal{O}_x \hat{\otimes} k'$

$\cong k'[[t]]$.

Ex 2.

$$I_x^+ \longrightarrow k \xrightarrow{\psi} \mathbb{C}^*$$

$$I_x^+(k') \longrightarrow k' \xrightarrow{Tr} k \xrightarrow{\psi} \mathbb{C}^*$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto b + \frac{c}{\tau} \mod \tau$$

Characters $\overset{}{\rightsquigarrow}_{\text{geom}}$ char. sheaves

(rk 1 local systems on K_x

$$\begin{array}{ccc} (K_S, \chi_S) & \rightsquigarrow & (K'_S, \chi'_S) \\ /k & \xrightarrow{\text{b-c}} & /k' \end{array}$$

$$A_c(k'; K'_S, \chi'_S)$$

<u>Def</u> (K_S, χ_S) is <u>weakly rigid</u> if $\dim A_c(k'; K'_S, \chi'_S)$ is uniformly bounded $\forall\, k'/k$.

<u>Relevant points</u>.

$$G(F)\backslash G(\mathbb{A})/K \longleftrightarrow \mathrm{Bun}_G(K)(k)$$

$f \in A_c(k; K_S, \chi_S)$ are functions on $\mathrm{Bun}_G(K_S^+)(k)$

where $K^+_x \lhd K_x$ $(x \in S)$ (9)

s.t. $\chi_x|K^+_x = 1$.

$K_x/K^+_x = k$-points of a finite dim'l gp $\underline{L_x}$.

Ex 1. $I^+_x \lhd \underset{\substack{\| \\ K_x}}{I_x} \longrightarrow \mathbb{G}_m(k)$

Ex 2. $\underset{\substack{\| \\ K^+_\infty}}{I^{++}_\infty} \lhd \underset{\substack{\| \\ K_\infty}}{I^+_\infty} \longrightarrow \underline{k \oplus k}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(b, \frac{c}{\tau}\right) \bmod \tau$

$C\left(\underline{\mathrm{Bun}_G(K^+_S)}(k)\right) \circlearrowleft \prod_{x \in S} L_x(k)$

eigenfunctions w. eigenval $(\chi_x)_{x \in S}$

$\|$

$\mathcal{A}_c(K_S, \chi_S)$.

$$\underline{\underline{\mathrm{Aut}(\mathcal{E})}} \circlearrowright \{\cdot \ \mathcal{E}, \cdot, \cdot, \ldots, \cdot\} \subset \mathrm{Bun}_G(K_S^+) \circlearrowleft \underline{\underline{\prod L_x(k)}}$$

$$\mathrm{Aut}(\mathcal{E}) \circlearrowright \underset{\mathcal{E}}{\cdot} \in \mathrm{Bun}_G(K_S)$$

$$\Rightarrow \underset{\text{alg. gp}}{\mathrm{Aut}(\mathcal{E})} \xrightarrow{ev_{\mathcal{E}}} \prod_{x \in S} L_x(k)$$

up to conjugacy.

<u>Def</u>. A k-point $\mathcal{E} \in \mathrm{Bun}_G(K_S)(k)$ is <u>(K_S, χ_S)-relevant</u> if

$$ev_{\mathcal{E}}^*\left(\prod_{x \in S} \chi_x\right)\Big|_{\mathrm{Aut}(\mathcal{E})^{\circ}(k)} = 1.$$

Similarly, $\dfrac{\text{relevant } k'\text{-pts}}{\bar{k}\text{-pts}}$.

<u>Fact</u> $\dim A_c(k'; K_S, \chi_S)$

$\leq \#\ (K_S', \chi_S')$-relevant k'-points of $\mathrm{Bun}_G(K_S)$

<u>Cor.</u> (K_S, χ_S) is weakly rigid iff there are finitely many (K_S, χ_S)-relevant $\bar{k}$-points of $\mathrm{Bun}_G(K_S)$.

<u>Ex 1</u> $G = Sl_2$

$K_x = \underline{\underline{I_x}}$ $\quad x \in \{0, 1, \infty\} = S$.

$$\mathrm{Bun}_G(K_S)(k) = \left\{ \left(\underset{\substack{\uparrow \\ \mathrm{rk}\,2}}{\mathcal{V}},\ \iota : \Lambda^2 \mathcal{V} \simeq \mathcal{O}_X,\ \{ \ell_x \subset \mathcal{V}_x \}_{x \in S} \right) \right\}$$

$$I_x \longrightarrow \mathbb{G}_m \overset{=}{} {}^{L_x} \quad \forall x \in S$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a \bmod \tau$$

$$\mathcal{E} = (\mathcal{V}, \iota, l_0, l_1, l_\infty).$$

$$\mathrm{ev}: \mathrm{Aut}(\mathcal{E}) \longrightarrow \prod_{x \in S} \underline{\mathbb{G}_m}$$

$\cup$

$\gamma : \mathcal{V} \longrightarrow \mathcal{V}$

$\gamma_0 \circlearrowright \mathcal{V}_0$ preserving l_0.

$\rightsquigarrow$ scalar of $\gamma_0 \circlearrowright l_0$.

$$\prod_{x \in S} \chi_x \Big|_{\underline{\mathrm{Aut}(\mathcal{E})^\circ}} \overset{?}{=} 1.$$

& $\mathcal{V} = \mathcal{O}^2$.

$l_0, l_1, l_\infty \subset k^2$

in generic pos.

$\mathrm{Aut}(\mathcal{E}) \ldots \ldots = \{\pm 1\}$

relevant.

Other pts are irrelevant.

$$V = \overset{\lambda}{L} \oplus \overset{\lambda^{-1}}{L'}.$$

s.t. $l_x \in L_x$ or $l_x \in L'_x$.

$$\mathbb{G}_m \subset \mathrm{Aut}(V, \dots) \longrightarrow \prod_{x \in S} \mathbb{G}_m$$

$$\chi_0^{\pm 1} \chi_1^{\pm 1} \chi_\infty^{\pm 1} \neq 1$$

$$\Rightarrow \left(ev_\varepsilon^* \prod \chi_x\right)\Big|_{\mathbb{G}_m} \neq 1.$$