Setup.

- $k = \mathbb{F}_q$

  X = proj. sm. geom connected curve / k. (e.g. $\mathbb{P}^1$)

  $k(X) = F$

  $|X|$ = closed points on X

  $$\begin{array}{ccccc} x\ , & F_x & \supset & \mathcal{O}_x & \twoheadrightarrow k_x \\ & \| & & \wr\| & \\ & k_x((t_x)) & \supset & k_x[[t_x]] & \end{array}$$

- G/k split semisimple gp

  SL_n, PGL_n, Sp_{2n},
  G_2, E_8, ...

- $$\mathbb{A} = \prod_{x \in |X|}{}' F_x$$

$$G(\mathbb{A}) = \prod_{x \in |X|}{}' G(F_x)$$

(almost all components $\in G(\mathcal{O}_x)$)

<u>Level gps</u>

$$K = \prod_{x \in |X|} K_x$$

$K_x \subset G(F_x)$ compact open

almost all $K_x = G(\mathcal{O}_x)$.

$$K^{\natural} = \prod_{x \in |X|} G(\mathcal{O}_x)$$

- $A_K = C\left(G(F)\backslash G(\mathbb{A})/K, \mathbb{C}\right)$

$\circlearrowleft$

$$H_K = \left\{ K\backslash G(\mathbb{A})/K \xrightarrow{h} \mathbb{C} \right\}$$

compactly supp

Hecke algebra (unit $= 1_K$)

$f: G(F)\backslash G(\mathbb{A})/K \longrightarrow \mathbb{C}$

$h. \ K\backslash G(\mathbb{A})/K \longrightarrow \mathbb{C}$

$$(f * h)(\underset{G(\mathbb{A})}{\underset{\cap}{x}}) = \sum_{g \in G(\mathbb{A})/K} f(xg)\, h(g^{-1})$$

(finite sum)

Study A_K as an H_K-mod.

- $A_{K,c}$ = cpt supp fun $\in A_K$ 4

$$A_{K,\text{cusp}} = \left\{ f \in A_{K,c} \;\middle|\; \dim_{\mathbb{C}} H_K \cdot f < \infty \right\}$$

eigenforms f: for almost all x
($K_x = G(\mathcal{O}_x)$)

f is an eigenvector
under $H_{K_x} = C_c(K_x \backslash G(F_x) / K_x)$

Bun_G

$$G(F) \backslash G(\mathbb{A}) / K^4 \quad K^4 = \prod G(\mathcal{O}_x)$$

$$\updownarrow$$

G-bundles on X.

$G = GL_n$

G-bundles $\longleftrightarrow$ vector bundles of rk n

$\underline{\mathrm{Isom}}(\mathcal{O}^{\oplus n}, \mathcal{V}) \longleftarrow\!\shortmid \mathcal{V}$
$\circlearrowleft GL_n$

"principal GL_n-bundle over X"

Weil:

$$GL_n(F) \backslash \underbrace{GL_n(\mathbb{A}) / K^4}_{\text{stabilizers}} \longleftrightarrow \mathrm{Vec}_n(X)$$

stabilizers $\overset{\sim}{\longrightarrow}$ automorphi

$g = (g_x) \in GL_n(\mathbb{A})$

Assume $g_x = 1 \quad \forall \; x \neq x_0$.

$\Lambda_{x_0} = g_{x_0} \mathcal{O}_{x_0}^{\oplus n} \subset F_{x_0}^{\oplus n}$

$\mathcal{O}_{x_0}$-submod of rank n.

Then glue $\underline{\underline{\Lambda_{x_0}}}$ with $\underline{\underline{\mathcal{O}_{X \setminus x_0}^{\oplus n}}}$

$j : X \setminus x_0 \hookrightarrow X$.

$\underset{\text{affine}}{U} \subset X$.

$j_* \mathcal{O}_{X \setminus x_0}^{\oplus n}$

$U \longmapsto \Gamma(\underline{\underline{U \setminus x_0}}, \mathcal{O}^{\oplus n}) \cap \Lambda_{x_0}$

inside $\underline{\underline{F_{x_0}^{\oplus n}}}$

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$$g_{x_0} = \begin{pmatrix} t_{x_0} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \rightsquigarrow \underline{\mathcal{O}(-x_0) \oplus \mathcal{O}^{\oplus n-1}}$$

$$\mathrm{Vec}_n(X) \longrightarrow GL_n(F) \backslash GL_n(\mathbb{A}) / K^q$$

$\mathcal{V} \longmapsto \exists\ U \subset X$

st $\mathcal{V}|_U \simeq \mathcal{O}_U^{\oplus n}$

$\Lambda_x = \mathcal{V}|_{\mathrm{Spec}\,\mathcal{O}_x}$

$= g_x \mathcal{O}_x^{\oplus n}$

(g_x)

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general G (split)

$$G(F) \backslash G(\mathbb{A}) / K^q \longleftrightarrow \left\{ \begin{array}{c} G\text{-bundles} \\ \text{on } X \end{array} \right\}$$

$G = Sp_{2n}$ G-bundles are the same as

$(\mathcal{V}, \omega)$: $\mathcal{V}$: rk $2n$ v.b.

$\omega : \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{V} \longrightarrow \mathcal{O}_X$. sympl.

$\underline{\mathrm{Bun}_G(k)} = \{ G\text{-bundles on } X \}$

$\mathrm{Bun}_G(R) = \{ G\text{-bundles on } X \otimes_k R \}$

Bun_G : Artin stack.

<u>Ex</u> $X = \mathbb{P}^1$.

$\mathrm{Bun}_G(k)/\cong$

$G = Gl_n$

$Vec_n(\mathbb{P}^1)/\cong \quad \longleftrightarrow \quad (d_1 \geq d_2 \geq \dots \geq d_n)$

$d_i \in \mathbb{Z}$

$\mathcal{O}(d_1) \oplus \dots \oplus \mathcal{O}(d_n)$

In general, $T \subset G$ max torus, W.

$\mathrm{Bun}_{G, \mathbb{P}^1}(k)/\cong \quad \longleftrightarrow \quad X_*(T)/W$

A_{K^G} = functions on $Bun_G(k)$

~~Hec~~ H_K-action ?

<u>Ex.</u> $G = Gl_n$

$$h_x = \mathbb{1}_{K_x \cdot \begin{pmatrix} t_x & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} K_x} \in H_{K_x}.$$

$$f : Bun_G(k) \longrightarrow \mathbb{C}.$$

$$f * h_x : \underset{\substack{\cup \\ \mathcal{V}}}{Bun_G(k)} \longrightarrow \mathbb{C}.$$

$$(f * h_x)(\mathcal{V}) = \sum_{\substack{\mathcal{V} \xrightarrow{i} \mathcal{V}' \\ \operatorname{coker}(i) \cong k_x}} f(\mathcal{V}')$$

(elem. upper modif. of $\mathcal{V}$)

$$h_x = 1_{K_x \begin{pmatrix} t_x^{\lambda_1} & & \\ & \ddots & \\ & & t_x^{\lambda_n} \end{pmatrix} K_x}$$

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$(f * h_x)(v) = \sum_{v \overset{\dots\dots>}{\scriptstyle\lambda, x} v'} f(v')$$

Level Structures. Fix $x \in |X|$.

parahoric subgps $\subset \underset{\sim}{G(F_x)}$

Iwahori $\qquad I_x \subset G(\mathcal{O}_x)$

$$\begin{tikzcd} I_x \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G(\mathcal{O}_x) \arrow[d] \\ B(k_x) \arrow[r, phantom, "\subset"] & G(k_x) \end{tikzcd}$$

$G = GL_2$. $\quad I_x = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{c} a,b,c,d \in \mathcal{O}_x \\ c \in m_x \end{array} \right\}$

Analogy:

G/k	G/F_x
Borel sgp	Iwahori
parabolic subgps	parahoric
Dynkin	affine Dynkin

$G = Gl_n$.

- $$\Lambda_x \subset F_x^{\oplus n}$$

  $$\mathrm{Stab}_{G(F_x)}(\Lambda_x) = \{ g \in Gl_n(F_x) \mid g\Lambda_x = \Lambda_x \}$$

  $$\Lambda_x = \mathcal{O}_x^{\oplus n} \rightsquigarrow \underline{\underline{Gl_n(\mathcal{O}_x)}}$$

- $$\Lambda_0 \underset{1}{\subset} \Lambda_1 \qquad \Lambda_1/\Lambda_0 \simeq k_x$$

  $\mathrm{Stab}(\Lambda_0, \Lambda_1)$ parahoric.

- $$\Lambda_0 \underset{a_1}{\subset} \underline{\underline{\Lambda_{a_1}}} \underset{a_2}{\subset} \cdots \subset \Lambda_n \subset \Lambda_{n+a_1} \subset \cdots$$
  $$\Lambda_n = t_x^{-1}\Lambda_0 \subset t_x^{-1}\Lambda_{a_1} \cdots = \Lambda_{n+a_1}$$

$\mathrm{Stab}_{GL_n(F_x)}(\Lambda_\bullet)$

give all parahoric subgps in $GL_n(F_x)$.

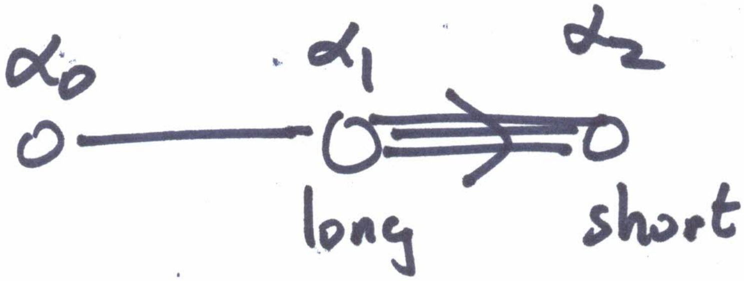
Iwahori:

$$\subset \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_n \subset \cdots$$

$$\Lambda_n = t^{-1}\Lambda_0$$

Affine Dynkin diagram.

G_2:



$\emptyset$: Iwahori.

$\{\alpha_1, \alpha_2\}$: $G(\mathcal{O}_x)$.

$\{\alpha_0, \alpha_1\}$: $P \twoheadrightarrow SL_3$

$$\{\alpha_0, \alpha_2\}: \quad Q \twoheadrightarrow SO_4$$

$$SO_4 = (Sl_2 \times Sl_2)/\Delta\{\pm 1\}$$