Arizona Winter School 2022 problem session: geometric aspects of automorphic forms

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Introduction

Sections 1 and 2 of this problem set are about background for the lectures of Ellen Eischen. Sections 3 and 4 are about background for the lectures of Zhiwei Yun. We will make frequent reference to the lecture notes written by both lecturers, as well as several other articles listed in the bibliography at the end of this document. (Warning: since the lecture notes are frequently updated, some reference numbers may be off.) This problem set is intended as a starting point for further exploration, not a comprehensive accounting of the theory invoked in either lecture. You are highly encouraged to note the parts you find most interesting and pursue further details in the corresponding references.

1 Double coset spaces for unitary groups

1.1 Unitary groups and their adelic points

Here we check a few properties of the group $U_{a,b}$ (also sometimes written $U(a,b)$) defined in Section 2.1 of [Eis22].

1. Check that $U_{a,b}(\mathbb{R})$ is compact if and only if either $a = 0$ or $b = 0$. (We may write U_a for $U_{a,0} \cong U_{0,a}$ for short.)

2. (a) Let G be any algebraic group over $\mathbb{Q}$ such that $G(\mathbb{Q}_v)$ is compact for some place v. Let

$$\mathbb{A}^v = \prod_{u \neq v}' \mathbb{Q}_u.$$

Prove that $G(\mathbb{Q})$ is discrete in $G(\mathbb{A}^v)$. You may use the fact that $\mathbb{Q}$ is discrete in $\mathbb{A}$.

   (b) Conclude that if $\mathscr{U}$ is a compact open subgroup of $U_a(\mathbb{A}^\infty)$, $gU_a(\mathbb{Q})g^{-1} \cap \mathscr{U}$ is finite for all $g \in U_a(\mathbb{A}^\infty)$.

   (c) Show that if $\mathscr{U}$ is neat in the sense of Remark 2.2.5 of [Eis22], then $gU_a(\mathbb{Q})g^{-1} \cap \mathscr{U}$ is trivial for all $g \in U_a(\mathbb{A}^\infty)$.

(d) Show that there exists an integer e_a depending only on a with the following property: $\mathscr{U}$ is neat whenever there is a prime l not dividing e_a such that the image of $\mathscr{U}$ in $U_a(\mathbb{Q}_l)$ is pro-l. (Hint: find e_a such that every root of unity that appears as an eigenvalue of an element of $U_a(\mathbb{Q})$ is an e_ath root of unity.) Reference: Proposition 4.1.1 of [Che04].

3. As usual, let K be a quadratic imaginary extension of totally real field K^+. We will prove the isomorphism

$$GU_{1,1} \cong (GL_2 \times \mathrm{res}_{K/K^+} \mathbb{G}_m)/\mathbb{G}_m$$

stated in Section 2.1.1 of [Eis22].

(a) As in Remark 2.1.4, one description of the K^+-points of $GU_{1,1}$ is

$$\left\{ g \in GL_2(K) \mid g^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{g} = \nu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ some } \nu \in (K^+)^\times \right\}.$$

Show that this condition on g is equivalent to

$$\overline{g} = \frac{\nu}{\det g} g.$$

(b) Use Hilbert's Theorem 90 to show that $(\det g)/\nu = \overline{\lambda}/\lambda$ for some $\lambda \in K^\times$ such that λg is defined over K^+.

(c) Show that the map $g \mapsto (\lambda g, \lambda)$ gives a bijection from $GU_{1,1}(K^+)$ to $(GL_2(K^+) \times K^\times)/(K^+)^\times$, and conclude.

4. Let K be a quadratic imaginary extension of a totally real field K^+, V an n-dimensional vector space over K, and $\langle , \rangle$ a nondegenerate K-valued Hermitian pairing on V. Check the statement in Section 2.1.2 that if v splits as $w\overline{w}$ in K, then $U(K_v^+) \cong GL_n(K_v^+)$.

1.2 Hermitian symmetric domains for unitary groups

We verify the assertions about Hermitian symmetric domains for unitary groups in Section 3.2 of [Lan22]. As in Lan's and Eischen's notation, let $a \geq b \geq 0$, and write $1_{a,b}$ or $I_{a,b}$ for the matrix $\begin{pmatrix} 1_a & \\ & -1_b \end{pmatrix}$ (i.e. the diagonal matrix with a 1s down the diagonal followed by b -1s. Then by definition

$$U_{a,b} = \{g \in GL_{a+b}(\mathbb{C}) \mid \overline{g}^T 1_{a,b} g = 1_{a,b}\}.$$

1. First we work through the details of the bounded realization $\mathcal{D}_{a,b}$.

(a) Lan writes

$$\mathcal{D}_{a,b} = \left\{ U \in M_{a,b}(\mathbb{C}) \mid \begin{pmatrix} \overline{U} & 1 \end{pmatrix} \begin{pmatrix} 1_a & \\ & -1_b \end{pmatrix} \begin{pmatrix} U \\ 1 \end{pmatrix} = \overline{U}^T U - 1_b < 0 \right\}$$

where $M_{a,b}$ is the space of $a \times b$ matrices and "$A < 0$" means that A is negative definite. Check Lan's assertion that given $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_{a,b}$, the map

$$U \mapsto gU = (AU + B)(CU + D)^{-1}$$

for $U \in \mathcal{D}_{a,b}$ gives a well-defined action of $U_{a,b}$ on $\mathcal{D}_{a,b}$. (Note that this action comes from the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U \\ 1 \end{pmatrix} = \begin{pmatrix} AU + B \\ CU + D \end{pmatrix} \sim \begin{pmatrix} (AU + B)(CU + D)^{-1} \\ 1 \end{pmatrix}$$

where, like Lan, we say for two rank-n $m \times n$ matrices X, Y that $X \sim Y$ if X can be multiplied on the right by an invertible $n \times n$ matrix to give Y. That is, $X \sim Y$ if X, Y give rise to the same point in the Grassmannian of n-dimensional subspaces of an m-dimensional vector space.)

(b) What is $\mathcal{D}_{1,1}$?

(c) Check Lan's assertion that if $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_{a,b}$ and $B = 0$, then also $C = 0$, and hence that the stabilizer of $0 \in \mathcal{D}_{a,b}$ is

$$\left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \in U_{a,b} \right\} \cong U_a \times U_b.$$

2. Now we go through the details of the unbounded realization $\mathcal{H}_{a,b}$. Like Lan, we define

$$U'_{a,b} = \{ g \in GL_{a+b}(\mathbb{C}) \mid \overline{g}^T J_{a,b} g = J_{a,b} \}$$

where

$$J_{a,b} = \begin{pmatrix} & & 1_b \\ & S & \\ -1_b & & \end{pmatrix}$$

where S is some choice of skew-Hermitian matrix satisfying $-iS > 0$ (i.e. $\overline{S}^T = -S$ and $-iS$ is positive definite). Let

$$\mathcal{H}_{a,b} = \left\{ \begin{pmatrix} Z \\ W \end{pmatrix} \in M_{a,b}(\mathbb{C}) \mid -i \begin{pmatrix} \overline{Z} & \overline{W} & 1 \end{pmatrix} J_{a,b} \begin{pmatrix} Z \\ W \\ 1 \end{pmatrix} = -i(\overline{Z}^T - Z + \overline{W}^T SW) < 0 \right\}$$

where Z is $b \times b$ and W is $(a - b) \times b$.

(a) Check that if $g = \begin{pmatrix} A & E & B \\ F & M & G \\ C & H & D \end{pmatrix} \in U'_{a,b}$, the map

$$\begin{pmatrix} Z \\ W \end{pmatrix} \mapsto g \begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} (AZ + EW + B)(CZ + HW + D)^{-1} \\ (FZ + MW + G)(CZ + HW + D)^{-1} \end{pmatrix}$$

gives a well-defined action of $U'_{a,b}$ on $\mathcal{H}_{a,b}$. Interpret this action as matrix multiplication on points of a Grassmannian as we did for $\mathcal{D}$.

(b) What is $\mathcal{H}_{1,1}$?

(c) For any $b \times b$ matrix Z, let

$$\Re(Z) = \frac{1}{2}(Z + \overline{Z}^T), \qquad \Im(Z) = \frac{1}{2i}(Z - \overline{Z}^T).$$

Note that these are *not* the usual real and imaginary parts of Z, but what Lan calls the "Hermitian" real and imaginary parts, in the sense that $\Re(Z)$ and $\Im(Z)$ are Hermitian and $Z = \Re(Z) + i\Im(Z)$. Show that

$$\mathcal{H}_{b,b} = \{Z \in M_b(\mathbb{C}) \mid \Im(Z) > 0\}.$$

(d) Let S be the skew-Hermitian matrix inside $J_{a,b}$ previously chosen. Then $-iS$ is Hermitian, and by the Spectral Theorem, there is $T \in GL_n(\mathbb{C})$ such that $T\overline{T}^T = -iS$. Show that

$$g \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}} & & -\frac{i}{\sqrt{2}} \\ & T & \\ \frac{1}{\sqrt{2}} & & \frac{i}{\sqrt{2}} \end{pmatrix} g \begin{pmatrix} \frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \\ & T^{-1} & \\ \frac{i}{\sqrt{2}} & & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

gives an isomorphism $U'_{a,b} \xrightarrow{\sim} U_{a,b}$, that

$$\begin{pmatrix} Z \\ W \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}} & & -\frac{i}{\sqrt{2}} \\ & T & \\ \frac{1}{\sqrt{2}} & & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} Z \\ W \\ 1 \end{pmatrix}$$

gives an isomorphism $\mathcal{H}_{a,b} \xrightarrow{\sim} \mathcal{D}_{a,b}$ taking $\begin{pmatrix} i1_b \\ 0 \end{pmatrix} \in \mathcal{H}_{a,b}$ to $0 \in \mathcal{D}_{a,b}$, and that these two isomorphisms are equivariant.

(e) Given any $\begin{pmatrix} Z \\ W \end{pmatrix} \in \mathcal{H}_{a,b}$, construct an element of $U'_{a,b}$ taking $\begin{pmatrix} Z \\ W \end{pmatrix}$ to $\begin{pmatrix} i1_b \\ 0 \end{pmatrix}$. Conclude that the actions of $U_{a,b}$ on $\mathcal{D}_{a,b}$ and $U'_{a,b}$ on $\mathcal{H}_{a,b}$ are transitive.

3. Finally, we modify these Hermitian symmetric domains to look like parts of Shimura data.

(a) Let

$$\begin{aligned} h_0 : U_1 &\to U_{a,b} \\ x + yi &\mapsto \begin{pmatrix} (x - yi)1_a & \\ & (x + yi)1_b \end{pmatrix}. \end{aligned}$$

Show that the centralizer of $h_0(U_1)$ in $U_{a,b}$ is $U_a \times U_b$, and conclude that $\mathcal{D}_{a,b}$ is isomorphic to the orbit of h_0 in $U_{a,b}$ under conjugation.

(b) Write out the analogous expression for $\mathcal{H}_{a,b}$ as the orbit of h'_0 in $U'_{a,b}$ under conjugation, for some $h'_0 : U_1 \to U'_{a,b}$.

(c) Let

$$GU_{a,b} = \{(g,r) \in GL_{a+b}(\mathbb{C}) \times \mathbb{R}^\times \mid \overline{g}^T 1_{a,b} g = r 1_{a,b}\}$$
$$GU'_{a,b} = \{(g,r) \in GL_{a+b}(\mathbb{C}) \times \mathbb{R}^\times \mid \overline{g}^T J_{a,b} g = r J_{a,b}\}.$$

Describe spaces $\mathcal{D}^{\pm}_{a,b}$ containing $\mathcal{D}_{a,b}$ and $\mathcal{H}^{\pm}_{a,b}$ containing $\mathcal{H}_{a,b}$, along with extensions of h_0 to $\mathbb{C}^\times \to GU_{a,b}$ and h'_0 to $\mathbb{C}^\times \to GU'_{a,b}$, so that $\mathcal{D}^{\pm}_{a,b}$ is the orbit of h_0 in $GU_{a,b}$ under conjugation and $\mathcal{H}^{\pm}_{a,b}$ is the orbit of h'_0 in $GU'_{a,b}$ under conjugation. (Note that Lan doesn't write these out, but he writes out the corresponding extension from Sp_{2n} to GSp_{2n} in Section 3.1.4.)

4. What happens when $b = 0$?

1.3 PEL data

1. Here we check the statements in Section 5.1.1 of [Lan22] translating PEL data into abelian varieties with Polarization, Endomorphism, and Level structure. Suppose we have an integral PEL datum $(\mathcal{O}, *, L, \langle,\rangle, h)$ as in Section 2.2 of [Eis22] or Section 5.1.1 of [Lan22].

(a) Let $V = L \otimes_{\mathbb{Z}} \mathbb{R}$, $A = V/L$, and

$$H : V \times V \to \mathbb{C}$$
$$(x,y) \mapsto \frac{1}{2\pi i}(\langle x,y\rangle - i\langle x, h(i)y\rangle).$$

As in Section 2.1 of [Lan12], check that h gives A the structure of a complex torus and that H is a positive definite Hermitian form such that $\Im(H)$ is integral on $L \times L$. It is standard that this data is necessary and sufficient to make A into an abelian variety.

(b) Let

$$L^{\#} = \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle x,y\rangle \in 2\pi i\mathbb{Z} \text{ for all } y \in L\}.$$

Let $A^\vee = (L \otimes_{\mathbb{Z}} \mathbb{R})/L^{\#}$. It turns out that $A^\vee$ is indeed the dual abelian variety of A and that the map $\lambda : A \to A^\vee$ induced by the natural inclusion $L \subset L^{\#}$ is a polarization of A. Following Section 2.2 of [Lan12], check as many details as you would like to convince yourself that λ really is a polarization.

(c) Describe a natural embedding $\iota : \mathcal{O} \to \mathrm{End}_{\mathbb{C}}(A)$ satisfying the Rosati condition that

$$\lambda \circ \iota(b^*) = (\iota(b))^\vee \circ \lambda$$

as maps $A \to A^\vee$.

(d) Check as many details as you would like to convince yourself that the natural isomorphism $L/nL \xrightarrow{\sim} A[n]$ gives rise to a principal level-n structure in the sense of Definition 1.3.6.1 of [Lan13].

2. Let K be a quadratic imaginary extension of a totally real field K^+. Let $V_1, \ldots, V_m$ be K-vector spaces equipped with Hermitian pairings $\langle,\rangle_{V_1}, \ldots, \langle,\rangle_{V_m}$. Check that the tuple $(D, *, \mathcal{O}_D, V, \langle,\rangle, L, h)$ described in Section 2.2.2 of [Eis22] (with $D = K^m$, etc.) really is a PEL datum of unitary type as defined in Section 2.2.

3. As in Example 5.1.3.5 of [Lan22], let $\mathcal{O}$ be an order in an imaginary quadratic extension E of $\mathbb{Q}$, with fixed isomorphism $E \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$. Let $*$ be complex conjugation of E over $\mathbb{Q}$. Let $L = \mathcal{O}^{a+b}$ with $a \geq b \geq 0$. Let $\epsilon \in \mathcal{O}$ be such that $-i\epsilon \in \mathbb{R}_{>0}$ and let

$$\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$$
$$(x, y) \mapsto \mathrm{tr}_{\mathcal{O}/\mathbb{Z}}(x^T J_{a,b} \overline{y})$$

where $J_{a,b} = \begin{pmatrix} & & 1_b \\ & \epsilon 1_{a-b} & \\ -1_b & & \end{pmatrix}$. Let G be the group over $\mathbb{Z}$ given by

$$G(R) = \{(g, r) \in \mathrm{End}_{\mathcal{O}}(L \otimes_{\mathbb{Z}} R) \times R^\times \mid \langle gx, gy \rangle = r\langle x, y \rangle \text{ for all } x, y \in L \otimes_{\mathbb{Z}} R\}$$

for each ring R. Let

$$h : \mathbb{C}^\times \to G(\mathbb{R})$$
$$re^{i\theta} \mapsto r \begin{pmatrix} \cos\theta & & -\sin\theta \\ & e^{-i\theta} & \\ \sin\theta & & \cos\theta \end{pmatrix}.$$

Check that h is well-defined, that $G(\mathbb{R}) \cong GU'_{a,b}$, and that the orbit of h in $G(\mathbb{R})$ under conjugation is $\mathcal{H}^{\pm}_{a,b}$.

2 Automorphic forms on unitary groups

2.1 Small cases

Here we explore the definitions of automorphic forms given in Section 3 of [Eis22] for small groups.

1. Recall that a Hecke character or Grössencharacter of a field K is a continuous group homomorphism
$$K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times.$$
   (a) Check that a Hecke character of $\mathbb{Q}$ is the same as an automorphic form on GL_1.
   (b) Check that if K is a quadratic imaginary extension of a totally real field, and GU_1 the corresponding general unitary group, a Hecke character of K is the same as an automorphic form on GU_1.

2. For the case of GL_2 (classical modular forms), write out as explicitly as possible the constructions of automorphic forms given in

(a) Definition 3.1.6 (as functions on a Hermitian symmetric domain).

(b) Definitions 3.2.6, 3.2.7, and 3.3.1 (as functions on spaces of abelian varieties). Also write out the correspondence in Lemma 3.2.2 in this particular case.

(c) Definition 3.4.2 (as functions on $GL_2(\mathbb{R})$.

(d) Definition 3.4.7 (as functions on $GL_2(\mathbb{A})$).

3. Repeat the previous problem for U_a.

4. For the case of GL_2, work through the details in Section 3.4.1 giving the bijection between automorphic forms defined as in Definition 3.1.6 (as functions on a Hermitian symmetric domain) and 3.4.7 (as functions on $GL_2(\mathbb{A})$). (Also see e.g. the introduction of [LW12].)

5. Use the isomorphism in Section 2.1.1 of [Eis22] (proven in Section 1.1, Problem 3 of this problem set) to explain the relationship between automorphic forms on $GU_{1,1}$ and forms on GL_2.

2.2 More general observations

We check some details in Section 3 of [Eis22].

1. Check the statement in Remark 3.1.5 that the defined automorphy factor $M_g(z)^T$ maps the lattice $p_{gz}(L)$ from Remark 2.2.8 to $p_z(L)$.

2. Check the last sentence in the proof of Lemma 3.2.2: that the map $f \mapsto F_f$ defined in the proof is well-defined and provides the inverse to the map $F \mapsto f_F$ in the statement of the Lemma.

3 Structure of Bun_G

3.1 An elementary proof of Grothendieck's theorem

In this section, we work through an elementary proof of Grothendieck's theorem that every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles $\bigoplus_i \mathscr{O}(n_i)$ for a unique multiset $\{n_i\}$. We follow [HM82].

1. First we reduce the statement to an elementary one about matrices.

   (a) Let k be any field and $\mathrm{spec}(k[s]) = \mathbb{A}^1_k \subset \mathbb{P}^1_k$. Let $\mathscr{E}$ be a vector bundle of rank m over $\mathbb{P}^1_k$. By looking at the transition functions of $\mathscr{E}$, explain why $\mathscr{E}$ is determined by an $m \times m$ matrix $A(s, s^{-1})$ which has coefficients in $k[s, s^{-1}]$.

   (b) Explain why we may assume that $\det(A(s, s^{-1}))$ is s^n for some $n \in \mathbb{Z}$.

(c) Show that $A(s, s^{-1})$ and $A'(s, s^{-1})$ give rise to isomorphic vector bundles iff there exist polynomial invertible $m \times m$ matrices $U(s), V(s^{-1})$ over $k[s]$ and $k[s^{-1}]$ respectively such that

$$A'(s, s^{-1}) = V(s^{-1})A(s, s^{-1})U(s).$$

(Note that we really mean that $U(s)$ is invertible in $M_{m\times m}(k[s])$, not just $M_{m\times m}(k[s, s^{-1}])$, and similarly for $V(s^{-1})$. In particular $\det(U(s))$ and $\det(V(s^{-1}))$ are nonzero constants.) In this case, we will say that $A(s, s^{-1})$ and $A'(s, s^{-1})$ are equivalent.

(d) Let $D(r_1, \ldots, r_m)$ be the diagonal matrix with diagonal entries $s^{r_1}, \ldots, s^{r_m}$. Describe the vector bundle associated to $A(s, s^{-1}) = D(r_1, \ldots, r_m)$. Hence argue that Grothendieck's theorem is equivalent to the following statement: for any $m \times m$ matrix $A(s, s^{-1})$ which has coefficients in $k[s, s^{-1}]$ and determinant equal to s^n for some $n \in \mathbb{Z}$, there exist polynomial invertible $m \times m$ matrices $U(s), V(s^{-1})$ over $k[s]$ and $k[s^{-1}]$ respectively such that

$$V(s^{-1})A(s, s^{-1})U(s) = D(r_1, \ldots, r_m)$$

with $r_1 \geq r_1 \geq \cdots \geq r_m \in \mathbb{Z}$, and the r_is are uniquely determined by $A(s, s^{-1})$.

2. Now we prove the uniqueness statement in 1(d).

(a) Suppose that $A(s, s^{-1})$ is equivalent (in the sense of 1(c)) to both $D(r_1, \ldots, r_m)$ and $D(r'_1, \ldots, r'_m)$. Observe that then we could find polynomial matrices with constant nonzero determinant $U(s), V(s^{-1})$ such that

$$V(s^{-1})D(r_1, \ldots, r_m) = D(r'_1, \ldots, r'_m)U(s).$$

For a matrix A, let $A^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$ be the minor of A given by the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$. The *Cauchy-Binet formula* says that

$$(AB)^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} = \sum_{r_1<\cdots<r_k} A^{i_1,\ldots,i_k}_{r_1,\ldots,r_k} B^{r_1,\ldots,r_k}_{j_1,\ldots,j_k}.$$

Using this, find some $i_1 < \cdots < i_k$ such that $r'_1 + \cdots + r'_k \leq r_{i_1} + \cdots + r_{i_k}$.

(b) Using the conclusion of 2(a), prove that $r_i = r'_i$ for all i.

3. Finally, we prove the existence statement in 1(d). We proceed by induction.

(a) Choose n so that $s^n A(s, s^{-1})$ is a polynomial matrix $B(s)$. Explain why we may assume that $B(s)$ satisfies $b_{11} = s^{k_1}$ for some $k_1 \in \mathbb{Z}_{\geq 0}$ and $b'_{1i} = 0$ for $i = 2, \ldots, m$.

(b) Assuming that the statement of 1(d) holds for $(m-1) \times (m-1)$ matrices, find matrices $U_1(s), V_1(s^{-1})$ satisfying the usual conditions so that $V_1(s^{-1})B(s)U_1(s)$ has the form

$$\begin{pmatrix} s^{k_1} & 0 & \cdots & 0 \\ c_2 & s^{k_2} & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots \\ c_m & 0 & \cdots & s^{k_m} \end{pmatrix}$$

where k_1 is the same as in 3(a), $k_2, \ldots, k_m \in \mathbb{Z}_{\geq 0}$, and $c_i \in k[s]$.

(c) Consider all matrices of the form in 3(b) which are equivalent to $B(s)$. Show that there exists one for which k_1 is maximal. Show that in this case, we must have $k_1 \geq k_i$ for $i = 2, \ldots, m$.

(d) Assuming as in 3(c) that $k_1 \geq k_i$ for $i = 2, \ldots, m$, explain how to eliminate $c_2, \ldots, c_m$ and complete the argument.

3.2 Weil's equivalence

This is a more detailed version of Exercise 1.2.2 in [Yun22]. We keep Yun's notation. As in Section 1.2.1 of [Yun22], let $G = GL_n$ and let $\mathrm{Vec}_n(X)$ be the groupoid of rank n vector bundles over X. Yun defines a map

$$e_S : \prod_{x \in S} G(F_x) \to \mathrm{Vec}_n(X)$$

for any finite subset $S \subset |X|$.

1. As in Exercise 1.2.2(1), show that the image of an element of $\prod_{x\in S} G(F_x)$ under e_S really is a vector bundle of rank n.

2. As in Exercise 1.2.2(2), combine the e_S as S varies to give a well-defined map $e : G(\mathbb{A}_F) \to \mathrm{Vec}_n(X)$.

3. As in Exercise 1.2.2(3), show that e is left invariant under $G(F)$ and right invariant under $K^\natural$.

4. Now we show that e is an equivalence of groupoids. First we check that the induced map $G(F)\backslash G(\mathbb{A}_F)/K^\natural \to \mathrm{Vec}_n(X)$ is a bijection of sets.

   (a) Check that this map is injective as follows: let $S \subset |X|$ be finite. Suppose that $(g_x), (g'_x) \in G(\mathbb{A}_F)$ are both nontrivial only at places in S and that $e_S((g_x)) = \mathscr{V}$, $e_S((g'_x)) = \mathscr{V}'$. If $f : \mathscr{V} \to \mathscr{V}'$ is an isomorphism, use f to find an element $h \in GL_n(\mathcal{O}_U) \subset GL_n(F)$, where $U = X \setminus S$, such that $(g'_x)^{-1} h g_x \in GL_n(\mathcal{O}_x)$ for all $x \in S$. Then use h to conclude that $(g_x), (g'_x)$ are in the same double coset.

   (b) Check that this map is surjective as follows: given a vector bundle $\mathscr{V}$ on X, find an open set $U \subset X$ on which $\mathscr{V}$ is trivial. Check that the stalk of $\mathscr{V}$ over any $x \in X \setminus U$ is of the form $g_x \mathcal{O}_x^{\oplus n}$ for some $g_x \in GL_n(F_x)$. Thus find an element of $G(\mathbb{A}_F)$ which maps to $\mathscr{V}$ under e.

   (c) Check that this map is an isomorphism of groupoids, with the following groupoid structure on $G(F)\backslash G(\mathbb{A}_F)/K^\natural$:

$$\mathrm{Hom}(g_1, g_2) = \{g_0 \in G(F) \mid g_0 g_1 \in g_2 K^\natural\} = G(F) \cap g_2 K^\natural g_1^{-1}.$$

3.3 Birkhoff decomposition

This is a more detailed version of Exercise 1.2.8 in [Yun22], proving a more general version of the results in Section 3.1 from a more conceptual viewpoint. We keep Yun's notation. We will construct a canonical bijection of sets

$$\mathrm{Bun}_G(k) \cong X_*(T)/W.$$

You can assume that $G = GL_n$ if you want.

1. As in Exercise 1.2.8(1), show that the map $X_*(T) \to T(\mathbb{A}_F)$ taking $\lambda \in X_*(T)$ to the element t^λ of $T(\mathbb{A}_F)$ which is $\lambda(t)$ at the place $0 \in \mathrm{spec}(k[t]) \subset \mathbb{P}^1$ and 1 elsewhere induces a bijection
$$X_*(T) \xrightarrow{\sim} T(F)\backslash T(\mathbb{A}_F)/K^\natural \cap T(\mathbb{A}_F).$$
Note that you can check this one diagonal coordinate at a time, since we are assuming that G is split.

2. As in Exercise 1.2.8(2), check that the map
$$T(F)\backslash T(\mathbb{A}_F)/K^\natural \cap T(\mathbb{A}_F) \to G(F)\backslash G(\mathbb{A}_F)/K^\natural$$
is W-invariant.

3. We will now show, as in Exercise 1.2.8(2), that this map is surjective.
   (a) Use the Chinese Remainder Theorem to show that $\mathbb{A}_F = F + K^\natural$ (i.e. every element of $\mathbb{A}_F$ can be written as the sum of an element of F and an element of $K^\natural$).
   (b) Recall from the notation in Section 1.1.2 of [Yun22] that B is a Borel subgroup containing T. Let N be the corresponding unipotent subgroup. Use $\mathbb{A}_F = F + K^\natural$ to show that $N(\mathbb{A}_F) = N(F)(N(\mathbb{A}_F) \cap K^\natural)$.
   (c) Using the Iwasawa decomposition for G, show that $T(F)\backslash T(\mathbb{A}_F)/K^\natural \cap T(\mathbb{A}_F) \to G(F)\backslash G(\mathbb{A}_F)/K^\natural$ is surjective.

4. Use Section 3.1, Exercise 2 to show that when $G = GL_n$, this map is injective. Then check (if you want) that injectivity holds for all G.

3.4 Miscellaneous

1. Do Exercise 1.2.5 in [Yun22] describing PGL_n-torsors over X in terms of line bundles.

2. Do Exercise 1.2.6 in [Yun22] describing Sp_{2n}- and SO_n- torsors over X in terms of line bundles.

3. If $\mathscr{E} = e((g_y)_{y\in|X|})$, let $\mathscr{E}(x) = e((g'_y)_{y\in|X|})$ where $g'_x = x^{-1}g_x$ and $g'_y = g_y$ for $y \neq x$. This is an example of a "modification" of a vector bundle. Prove the formula in Section 1.2.9 describing the action of the spherical Hecke algebra in terms of modifications of vector bundles. What is the analogy with the Hecke operator action on classical modular forms?

4 The affine Grassmannian

Let Gr_G be the affine Grassmannian of G, described briefly in Section 4.2 of [Yun22] and in more detail in Section 2 of [Zhu16]. As in our sources, define the k-presheaves LG and L^+G by

$$LG(R) = G(R((t)))$$
$$L^+G(R) = G(R[\![t]\!])$$

for any k-algebra R. Then we have $Gr_G = LG/L^+G$. L^+G acts on Gr_G by left translation and its orbits correspond to elements of

$$G(k[\![t]\!])\backslash G(k((t))/G(k[\![t]\!]) \cong X_*(T)/W \cong X_*(T)^+.$$

We write $D_R = \operatorname{spec} R[\![t]\!]$, $D_R^\times = \operatorname{spec} R((t))$. We often work with the data of two G-torsors $\mathscr{E}_1, \mathscr{E}_2$ over $D = D_k$ along with an isomorphism $\beta : \mathscr{E}_1|_{D^\times} \xrightarrow{\sim} \mathscr{E}_2|_{D^\times}$. This data is associated to an element

$$\operatorname{Inv}(\beta) \in G(k[\![t]\!])\backslash G(k((t))/G(k[\![t]\!]) \cong X_*(T)^+.$$

4.1 Schubert varieties are closed

In this section, we work through the proof of Proposition 2.1.4 in [Zhu16]. The statement is that for a given $(\mathscr{E}_1, \mathscr{E}_2, \beta)$, $X = \operatorname{spec} R$, and $\mu \in X_*(T)^+$, the set

$$X_{\leq\mu} = \{x \in X \mid \operatorname{Inv}(\beta_{k(x)}) \leq \mu\}$$

is Zariski-closed in X. We will focus on the case of $G = GL_n$, but you can think about more general G if you want.

1. For a given $(\mathscr{E}_1, \mathscr{E}_2, \beta)$, let $x \in X_{\leq\mu}$. Let $\rho_\chi : G \to GL(V_\chi)$ be a finite dimensional highest weight representation of G of highest weight χ, and for $i = 1, 2$ let $V_{\chi,\mathscr{E}_i} = \mathscr{E}_i \times^G V_\chi$. Explain why x satisfies
$$\rho_\chi(\beta_x)(V_{\chi,\mathscr{E}_1}) \subset t^{-\langle\chi,\mu\rangle}(V_{\chi,\mathscr{E}_2}).$$
2. Explain why the set $X_{V_\chi,\leq\mu}$ of those x satisfying the containment in 1 is Zariski closed.
3. In the case $G = GL_n$, show that
$$X_{\leq\mu} = \bigcap_{V_\chi} X_{V_\chi,\leq\mu}$$
where the intersection is taken over all finite dimensional highest weight representations of G. Conclude that $X_{\leq\mu}$ is closed. (Optional: think about, or look up, how to conclude for more general G.)

4.2 L^+G-orbits

In this section, we work through the proof of Proposition 2.1.5 in [Zhu16]. Let

$$Gr_\mu = Gr_{\leq\mu} \setminus \bigcup_{\lambda<\mu} Gr_{\leq\lambda}.$$

1. Show that Gr_μ forms a single L^+G-orbit.
2. Show that Gr_μ is smooth.
3. Show that the tangent space of Gr_μ at t^μ can be written in the form

$$\bigoplus_{\langle\alpha,\mu\rangle\geq 0} \mathfrak{g}_\alpha(\mathcal{O})/t^{\langle\alpha,\mu\rangle}\mathfrak{g}_\alpha(\mathcal{O}),$$

   where α ranges over all positive roots of G and $\mathfrak{g}_\alpha$ is the root space corresponding to α. Conclude that $\dim Gr_\mu = \langle 2\rho, \mu\rangle$ where 2ρ is the sum of the positive roots of G.

4. Now we show that if $\lambda \leq \mu$, then Gr_λ is contained in the Zariski closure of Gr_μ. Let α be a positive coroot such that $\mu - \alpha$ is dominant and $\lambda \leq \mu - \alpha \leq \mu$ (such a coroot always exists).

   (a) Explain why it suffices to construct a curve C in $Gr_{\leq\mu}$ such that $t^{\mu-\alpha} \in C$ and $C \setminus \{t^{\mu-\alpha}\} \subset Gr_\mu$.

   (b) For any integer m, let $t^{\lambda_m} = \begin{pmatrix} t^m & 0 \\ 0 & 1 \end{pmatrix}$ and

$$K_m = \mathrm{Ad}_{t^{\lambda_m}}(L^+SL_2) \subset LSL_2.$$

   Let $i_\alpha : SL_2 \to G$ be the homomorphism corresponding to the coroot α and $m = \langle \mu, \alpha\rangle - 1$. Let

$$C_{\mu,\alpha} = Li_\alpha(K_m)t^\mu.$$

   Check that $C_{\mu,\alpha} \cong \mathbb{P}^1$.

   (c) Check that $C_{\mu,\alpha}$ contains $t^{\mu-\alpha}$ and that $C \setminus \{t^{\mu-\alpha}\} \subset Gr_\mu$.

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