AWS 2022: MODULAR FORMS ON EXCEPTIONAL GROUPS

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1. Introduction

Denote by $\mathfrak{h}$ the upper half plane in $\mathbf{C}$. Classical modular forms are holomorphic functions $f : \mathfrak{h} \to \mathbf{C}$ that satisfy a functional equation for congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$ and a moderate growth condition. Associated to a classical modular form f is an automorphic function $\varphi_f : \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$. It is fruitful to think of classical modular forms as automorphic functions φ on $\mathrm{GL}_2(\mathbf{A})$ that give rise to holomorphic functions on the symmetric space $\mathfrak{h}$ associated to $\mathrm{PGL}_2(\mathbf{R})$.

If G is a reductive group over $\mathbf{Q}$ for which the symmetric space $X_G = G(\mathbf{R})/K$ associated to G has the structure of a complex manifold, then one can talk about holomorphic modular forms on X_G. Equivalently, one can define a special class of automorhpic functions for G that correspond to these holomorphic modular forms. These special automorphic functions are distinguished among all automorphic forms. For one, they have a classical Fourier or Fourier-Jacobi expansion with semi-classical Fourier coefficients; one can extract arithmetic information from the Fourier coefficients in a way that cannot necessarily be mimicked for general automorphic forms. For another, these special holomorphic modular forms (or the associated automorphic forms) tend to show up in the applications of automorphic forms to special values of L-functions.

Suppose now G is reductive $\mathbf{Q}$-group for which the symmetric space X_G does not have complex structure. One can ask if there exists a class of automorphic functions on $G(\mathbf{A})$ that can take the place of the holomorphic modular forms. For example, if G is of Dynkin type G_2, F_4 or E_8 then *no real form* of G has a symmetric space with complex structure, so this is a relevant question if you wish to produce some analogue of classical modular forms to G_2 or E_8.

Fortunately, the question as to whether there are good analogues of holomorphic modular forms appears to have an answer. For certain real forms of the exceptional algebraic $\mathbf{Q}$-groups, such as split G_2, split F_4, and $E_{8,4}$, Gross-Wallach [GW94, GW96] and Gan-Gross-Savin [GGS02] singled out a special class automorphic forms on these groups that has the potential to be an analogue of classical holomorphic modular forms. The groups in question are called the quaternionic exceptional groups, and we emphasize that their associated symmetric space does not have complex structure.

Let us call the special automorphic forms on these quaternionic exceptional groups the *quaternionic modular forms*. This course is about the quaternionic exceptional groups and the associated quaternionic modular forms. In particular, I will define these objects and present evidence, largely from [Pol20a], [Pol20b], [Pol21], that quaternionic modular forms behave like classical holomorphic modular forms.

2. Course Outline

Here is tentative outline of the lectures in this course.

- **Lecture 1**: How to work with exceptional groups: Cubic norm structures, the Freudenthal construction, and explicit constructions of the exceptional Lie algebras.
- **Lecture 2**: Overview: What are the quaternionic exceptional groups. What is the definition of quaternionic modular forms (QMFs). What is known and not known about QMFs.

- **Lecture 3**: The Fourier expansion of QMFs: What is it, and how do you prove it. Applications of the Fourier expansion, such as the fact that cusp forms only have non-degenerate Fourier coefficients.
- **Lecture 4**: Some special modular forms: Cusp forms on G_2 with algebraic Fourier coefficients; modular forms (e.g. on E_7 and E_6) whose Fourier coefficients bear some relationship to arithmetic invariant theory.

3. Project outlines

While the course will focus mostly on exceptional groups, it is important to note that the classical groups $O(4,n)$ and $U(2,n)$ also support quaternionic modular forms. While the symmetric space for $U(2,n)$ does have complex structure, the quaternionic modular forms on this group do not correspond to holomorphic modular forms. The goal of the projects is to develop some of the theory of QMFs in the setting of the classical group $U(2,n)$.

- **Project 1**: The first project is to give the precise form of the Fourier expansion of QMFs on the group $U(2,n)$. The case of $U(2,n)$ was excluded from [Pol20a] because it could not be handled by the same uniform argument that handled $O(4,n)$ and the quaternionic exceptional groups. However, a direct argument should yield the form of the Fourier expansion on these groups. From the Fourier expansion, one should also be able to prove that cusp forms only have non-degenerate Fourier coefficients, and (from the proof of the Fourier expansion) produce examples of modular forms via Eisenstein series.
- **Project 2**: The second project is to give an analogue of the Oda-Rallis-Schiffman lift to $U(2,n)$. The Oda-Rallis-Schiffman lift is a theta lift from holomorphic modular forms f on SL_2 or its double cover to holomorphic modular forms $\theta(f)$ on the group $O(2,n)$. It is a very special instance of the general theta lift from symplectic to orthogonal groups, but in this special case one can write simple formulas for the Fourier coefficients of $\theta(f)$ in terms of those of f. The goal of this project is produce an analogue of this lift from holomorphic modular forms g on $U(1,1)$ to quaternionic modular forms $\theta(g)$ on $U(2,n)$, and to give simple formulas for the Fourier coefficients of $\theta(g)$ in terms of those of g. Note that this project assumes a successful completion of Project 1, but can mostly be worked on independently of that project. Also note that it would be a good idea to pay close attention to Wee Teck Gan's lectures on the theta correspondence to work on this project.

References

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