Lecture 3: · Examples of MFs on G_2

· Thms about MFs on G_2

Degenerate Eis series

Recall: $G_2 \supseteq P = MN$, $M \simeq GL_2$

Heis parabolic

$$\nu : P \to M \xrightarrow{\det} GL_1$$

· Suppose $\ell > 0$ even

Consider $\mathrm{Ind}_P^G(|\nu|^s)$

Let $f_{\ell,\infty}(g,s) \in \mathrm{Ind}_{P(\mathbb{R})}^{G_2(\mathbb{R})}(|\nu|^s) \otimes \mathbb{V}_\ell$

Recall: $\mathbb{V}_\ell = Sym^{2\ell}(\mathbb{C}^2) \boxtimes \mathbb{1} \circlearrowleft K \leq G_2$

be defined by:

- $f_{\ell,\infty}(pg;s) = |N(p)|^s f_{\ell,\infty}(g;s) \;\forall\, p \in P(\mathbb{R})$
- $f_{\ell,\infty}(gk;s) = k^{-1} \cdot f_{\ell,\infty}(g) \;\forall\, k \in K$

By Iwasawa decomp: $G_2(\mathbb{R}) = P(\mathbb{R})K$,

$f_{\ell,\infty}$ uniquely determined once we set

$$f_{\ell,\infty}(1) = x^\ell y^\ell \in \mathbb{V}_\ell = \langle x^{2\ell}, x^{2\ell-1}, \dots, y^{2\ell} \rangle$$

Let $f_{fte}(g,s) \in \mathrm{Ind}_{P(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(|\nu|^s)$ be a

flat section, i.e. $f_{fte}\big|_{G_2(\hat{\mathbb{Z}})}$ is ind. of s

Let $\underset{\substack{\in \\ G_2(\mathbb{A})}}{f_\ell(g,s)} = f_{fte}(g_f,s)\, f_{\ell,\infty}(g_\infty;s)$

Define

$$E_\ell(g,f,s) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G_2(\mathbb{Q})} f_\ell(\gamma g, s)$$

Conv: If $\mathrm{Re}(s) > 3$

$E_\ell(g) := E_\ell(g, f, s = \ell+1)$

Thm: If $\ell > 0$ is even & $\ell+1 > 3$ (ie $\ell \geq 4$) then $E_\ell(g)$ is a QMF on G_2 of wt ℓ.

Proof: . $f_{\ell,\infty}(g, s = \ell+1) \xrightarrow{D_\ell} 0$

. $E_\ell(g)$ is annihilated by D_ℓ b/c $\sum$ conv. absolutely

<u>Rmk:</u>

- π cusp aut repn of $GL_2 = M$ assoc. to a Hol wt 3ℓ mod form, cuspidal,
- $f_\pi \in \mathrm{Ind}_{P(\mathbb{A})}^{G_2(\mathbb{A})}(\pi)$
- $E(g, f_\pi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G_2(\mathbb{Q})} f_\pi(\gamma g)$
- If $\ell \geq 6$, this is a wt ℓ mod form on G_2
- If $\ell = 4$, can still make sense of $E(g, f_\pi) \Rightarrow$ wt 4 MF on G_2 assoc. $\triangle$

FACT 1: If φ is a level 1 QMF on G_2 of wt ℓ

- $a_\varphi(w) \neq 0 \implies f_w(u,v)$ is integral, ie
  $= au^3 + \dots + dv^3$, $a,b,c,d \in \mathbb{Z}$

- $$\boxed{a_\varphi(w \cdot \gamma) = \det(\gamma)^\ell a_\varphi(w)}$$
  $\gamma \in GL_2(\mathbb{Z})$

FACT: 2 There is a canonical bijection

$$(\text{Int. BCFs})/_{GL_2(\mathbb{Z})} \longleftrightarrow (\text{Cubic Rings})/_{\text{isom}}$$

THUS: If $\ell > 0$ is even, φ a level 1 wt ℓ MF on G_2, A cubic ring, can define

$$a_\varphi(A) = a_\varphi(w) \text{ if}$$

$$A \longleftrightarrow f_w$$

Rmk: If f_w is non-deg,
the cubic ring assoc. to f_w $A(f_w)$
is totally real $\Longleftrightarrow$ f_w is pos semi defin.

Thm (Gan-Gross-Savin, Jiang-Rallis) Suppose
A is the max'l order in a T.R. cubic étale $\mathbb{Q}$-alg.
E. There exists $c_\ell \in \mathbb{C}$ (indep of A) s.th

$$a_{E_\ell}(A) = c_\ell \, \zeta_E(1-\ell), \quad \ell \text{ even}.$$

Rmk: c_ℓ is not known to be nonzero

Open Question: $E(g, f_\pi) \leftarrow$ Eis series
Can anything be said about these F.C.'s?

Cusp forms

<u>Thm</u> Suppose $\ell \geq 16$ is even. There exist nonzero cusp forms on G_2 of wt ℓ, all of whose F.C.'s are alg integers

Pf of Thm

1 Start w/ Hol SMF f on $Sp(4)$ of wt ℓ. f has F.C.'s $\in \mathbb{Z}$.

2 Θ-lift f to $SO(4,4)$; obtain $\Theta(f)$.

$$\Theta(f)(g) = \int_{[Sp(4)]} \widetilde{\Theta(g,h)}\, \overline{f(h)}\, dh \quad \text{where}$$

Θ on $SO(4,4) \times Sp(4)$ is a Θ-fcn

(3) There exists a notion of QMF on the gp $SO(4,4)$. Can choose $\mathcal{E}(g,h)$ s.t. $\mathcal{E}(f)$ is a QMF on $SO(4,4)$ of wt ℓ.

(4) One can express the F.C.'s of $\mathcal{E}(f)$ in terms of the classical F.C.s of f

<u>In particular:</u> F.C. of $\mathcal{E}(f) \in \overline{\mathbb{Z}}$

(5) $G_2 \overset{i}{\hookrightarrow} SO(4,4)$. $i^*(\mathcal{E}(f))$: QM on G_2

<u>Pull back:</u> - Is still cuspidal

- Still has F.C.s $\in \overline{\mathbb{Z}}$

In fact, in wt 20, $\exists \neq 0$ cusp MF on G_2 w/ all F.C.s $\in \mathbb{Z}$.

Thm (R. Dalal) There exists an explicit dim^n formula for the level 1, cuspidal, QMFs on G_2 of wt ℓ. In particular, the smallest $\neq 0$ level 1 cusp form is in wt 6.

Thm (Cicek - Davidoff, Dijols, Hammonds, P, Roy)
Suppose φ is a level 1 cusp QMF on G_2 associated to a cusp aut rep^n π on $G_2(\mathbb{A})$. Suppose moreover $a_\varphi(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}) \neq 0$. Then

1. The completed std L-fcn has fcnl eq^n:

$$\Lambda(\pi, \mathrm{Std}, s) = \Lambda(\pi, \mathrm{Std}, 1-s)$$

(2) $\exists$ a Dir series for this L-fcn:

$$\sum_{\substack{T \subseteq \mathbb{Z}^3 \\ n \geq 1}} \frac{a_\varphi(\mathbb{Z} + nT)}{[\mathbb{Z}^3 : T]^{s-\ell+1} n^s}$$

$$= a_\varphi(\mathbb{Z}^3) \frac{L(\pi, \mathrm{Std}, s-2\ell+1)}{\zeta(s-2\ell+2)^2 \, \zeta(2s-4\ell+2)}$$

<u>Pf:</u> Make a refined analysis of a Rankin-Selberg integral due to Gurevich-Segal.

Half-integral wt

<u>Thm</u>. (Leslie-P.) $\exists$ a theory of half-integral wt MFs on G_2. These also have a good notion of F.C.'s, which are elts of $\mathbb{C}/\pm 1$.

- Suppose R is a cubic ring $\subseteq E$, TR cubic field
- $c_R \approx$ sq roots of ∂_R^{-1} in the narrow class gp of E.

<u>Precisely</u>: Say (I, μ) is <u>balanced</u> if

- I fractional R-ideal
- $\mu \in E_{>0}^{\times}$ tot pos.
- $I\mu^2 \subseteq \partial_R^{-1}$
- $N(I)^2 N(\mu) \operatorname{disc}(R) = 1$

If R is the max'l order: (I, μ) balanced

$$(-) \quad I^2\mu = \partial_R^{-1}.$$

$(I, \mu) \sim (I', \mu')$ if $\exists\ \beta \in E^\times$ s.t.

$$I' = \beta I, \quad \mu' = \beta^{-2}\mu$$

$$\left((I')^2 \mu' = I^2 \mu \right)$$

$$G_R = \{(I,\mu) \text{ balanced}\}/\sim.$$

<u>Rmk:</u> 1) Inspired by work of A. Swaminathan

2) G_R can be empty

3) If G_R is nonempty, R max'l order in E, then

$$|G_R| = 4 \cdot \# \mathrm{Cl}_E^+[2]$$

<u>Thm</u> (Leslie–P.) $\exists$ a wt 1/2 MF Θ' on G_2 whose F.C.s include the #s $\pm|\mathrm{Cl}_R|$ for R <u>even monogenic</u>, i.e.

$$R \simeq \mathbb{Z}[y]/(y^3+cy^2+by+a), \quad \text{w/ } a,b,c \in 2\mathbb{Z}.$$

<u>Pf:</u> $\cdot\ \Theta$ on $\widetilde{F}_4$

$\cdot\ \Theta' =$ pullback to $\widetilde{G}_2 \hookrightarrow \widetilde{F}_4$. $\blacksquare$