Lecture 2

Last time: G_2, MFs on G_2

Rmk: $\cdot$ $G_2/K = X$: not have a G_2-invt $\mathbb{C}$ structure

$\cdot$ $SL_2(\mathbb{R})/SO(2) = \mathfrak{h} \leftarrow SL_2(\mathbb{R})$-invt $\mathbb{C}$ structure

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Today: F.E. & Mod forms on G_2

$$f(z) = \sum_{n \geq 0} a_f(n) q^n, \quad \text{wt } \ell$$

Recall: $\phi_f(g) = j(g,i)^{-\ell} f(g \cdot i)$

$$\phi_f : Sl_2(\mathbb{R}) \to \mathbb{C}$$

<u>Define:</u> $W_n(g) : SL_2(\mathbb{R}) \to \mathbb{C}$ as

$$W_n(g) = j(g,i)^{-\ell} e^{2\pi i n (g\cdot i)}$$

<u>Properties</u>

- $W_n\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = e^{2\pi i n x} W_n(g)$

- $W_n(g k_\theta) = e^{-i\ell\theta} W_n(g)$

$$k_\theta = \begin{pmatrix} \cos(\theta) & -\sin\theta \\ \sin\theta & \cos(\theta) \end{pmatrix}$$

- $X_- W_n \equiv 0$

$\times$ $\cdot$ $W_n\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} = y^{\ell/2} e^{-2\pi n y}$ is completely explicit

$\phi_f(g) = \sum_{n\geq 0} a_f(n) W_n(g)$ is the F.E.

<u>What will happen:</u> $\varphi : {}_{\Gamma}\backslash G_2 \to \mathbb{V}_\ell$

$\mathbb{V}_\ell = \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \otimes \mathbb{1} \circlearrowleft K = SU(2) \times SU(2) / \pm 1$

$$\varphi(g) \text{ "=" } \sum_{\substack{f \\ \text{index set}}} a_\varphi(f)\, W_f(g)$$

= binary cubics

where $a_\varphi(f) \in \mathbb{C}$: the F.C.'s

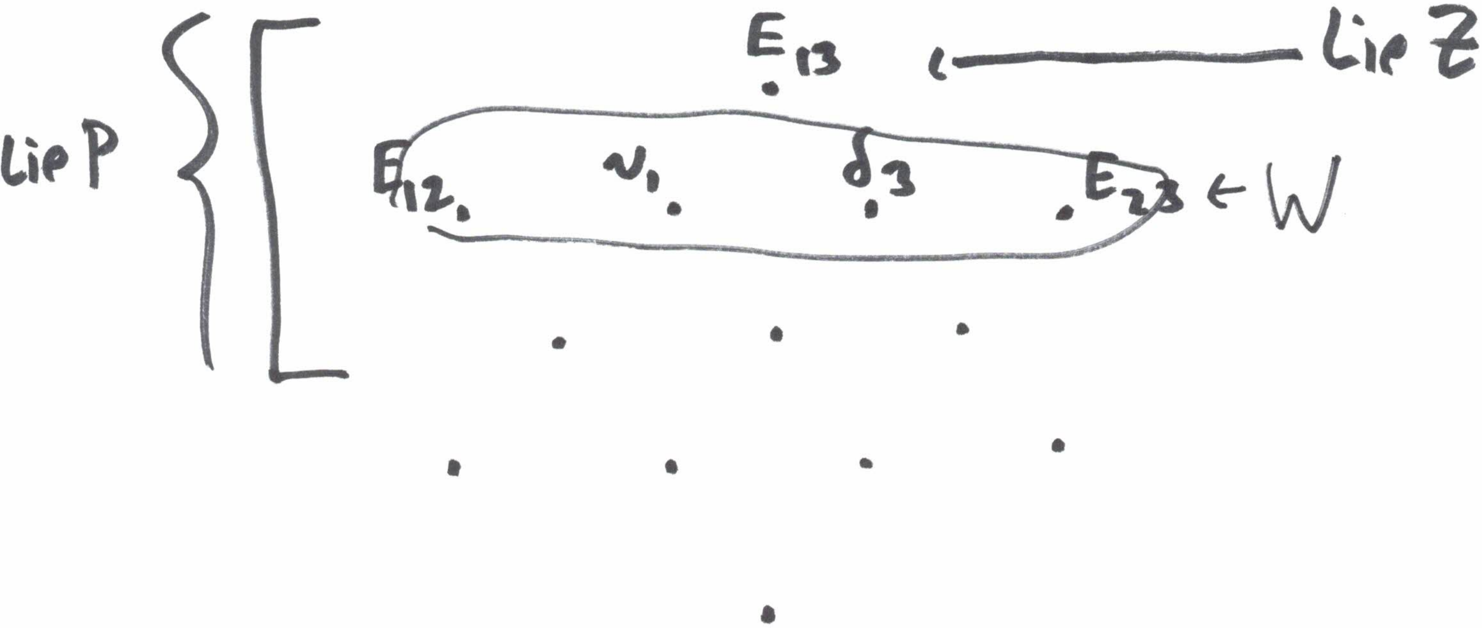
W_f satisfies similar properties

$\cdot$

$\cdot$

<u>Recall</u>: $\mathfrak{g}_2 = \mathfrak{sl}_3 + V_3 + V_3^\vee$

$\{E_{ij}\}$ $\quad v_1, v_2, v_3 \quad$ $\delta_1, \delta_2, \delta_3$



- G_2 has 2 conj. classes of max'l parab. subgps

- Let P be the parabolic w/ Lie alg as above.

- $P = MN$: $\quad M \simeq GL_2$

$N \supseteq Z = [N, N]$ and

N/Z is abln.

$$Z = \exp(\mathbb{R}E_{13})$$

$$N/Z = \exp(\underbrace{\mathbb{R}E_{12} + \mathbb{R}v_1 + \mathbb{R}\delta_3 + \mathbb{R}E_{23}}_{W})$$

$M \circlearrowright Z$ as $\det$

$M \circlearrowright N/Z$ as $\mathrm{Sym}^3(V_2) \otimes \det(V_2)^{-1}$.

There is a symplectic form on W

$$\langle\, ,\rangle : W \times W \to \mathbb{R} \quad \text{as}$$

$$[w, w'] = \langle w, w' \rangle E_{13}$$

<u>Explicitly</u>: $w = aE_{12} + \frac{b}{3}v_1 + \frac{c}{3}\delta_3 + dE_{13}$

$w' = $ — — — —

$$\langle w, w' \rangle = ad' - \frac{bc'}{3} + \frac{cb'}{3} - da'$$

$$\langle mw, mw' \rangle = \det(m) \langle w, w' \rangle$$

Chars of N Suppose

φ: is an aut form on $G_2(\mathbb{A})$

ψ: fixed add'l char $\mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$

$w \in W(\mathbb{Q})$

Define: $\varphi_w(g) = \displaystyle\int_{[N]} \psi^{-1}(\langle w, \bar{n}\rangle)\, \varphi(ng)\, dn$

where $\bar{n}$ denotes image of n in $N/Z \overset{\exp}{=} W$

$\varphi_Z(g) = \displaystyle\int_{[Z]} \varphi(zg)\, dz$, $\varphi_N(g) = \displaystyle\int_{[N]} \varphi(ng)\, dn$

THEN

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \in W(\mathbb{Q})} \varphi_w(g)$$

Will produce a refinement

Prop^n If $\varphi_Z(g) \equiv 0$ then $\varphi(g) \equiv 0$

Suppose $\varphi : \underset{G_2(\mathbb{Q})}{G_2(\mathbb{A})} \backslash \to \mathbb{V}_\ell$ is a mod form of wt ℓ

The fcns φ_w satisfy:

(0) $\varphi_w(g)$ is of mod growth

(1) $\varphi_w(ng) = \psi(\langle w, \bar{n} \rangle)\, \varphi_w(g)$

(2) $\varphi_w(gk) = k^{-1} \varphi_w(g)$

(3) $D_\ell \varphi_w \equiv 0$

Def^n Call a fcn F that satisfies these prop. a Generalized Whittaker fcn of type (w, ℓ).

<u>Will state Thm</u> Fcn satisfying

(0) — (3) are uniquely determined

up to scalar multiple.

$$\varphi_w(g) = \lambda W_w(g) \quad \text{for}$$

some explicit W_w

$\Longrightarrow$

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \neq 0} a_\varphi(w) W_w(g)$$

for φ a MF of wt l.

Identify W = Binary cubics

as

$$w = aE_{12} + \frac{b}{3}\gamma_1 + \frac{c}{3}\delta_3 + dE_{23}$$

$$\longmapsto au^3 + bu^2v + cuv^2 + dv^3 = f_w$$

<u>Define:</u> $w \neq 0$, $w \in W(\mathbb{R})$.

$$\beta_w(\underset{\substack{\in \\ GL_2(\mathbb{R})}}{m}) = \langle w, m \cdot (u - iv)^3 \rangle$$

These appear in F.E. on G_2

<u>Prop$^{\underline{n}}$:</u> TFAE

1) $\beta_w(m) \neq 0 \quad \forall\, m \in GL_2(\mathbb{R})$
2) $f_w(z, 1) \neq 0$ on $\mathfrak{h} \ni z$
3) f_w splits into lin factors $/\mathbb{R}$

If w satisfies these props, say w is pos. semi-definite and write $w \geq 0$.

<u>Ex:</u> (1) $f_w(u,v) = au^3 \geq 0$

(2) $f_w(u,v) = -u^3 + uv^2 = u(v-u)(v+u)$

≥ 0

(3) $u^3 + v^3 \not\geq 0$

Define, for $m \in GL_2(\mathbb{R}) = M(\mathbb{R})$

$$W_w(m) = |\det(m)| \det(m)^{\ell} \times$$

$$\sum_{-\ell \leq \nu \leq 0} \left(\frac{|\beta_w(m)|}{\beta_w(m)} \right)^{\nu} K_\nu(|\beta_w(m)|) \frac{x^{\ell+\nu} \, y^{\ell-\cdots}}{(\ell+\nu)!\,(\ell-\cdots}$$

for $w \geq 0$.

- $\{x^{2\ell}, x^{2\ell-1}y, \dots, y^{2\ell}\}$ ← basis of V_ℓ

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}\, t^\nu \,\frac{dt}{t} \text{ is}$$

the K-Bessel fcn

<u>Rmk:</u> K_ν diverges at 0

These fcns $W_w : M(\mathbb{R}) \to \mathbb{V}_\ell$ extend uniquely to

$$G_2 \to \mathbb{V}_\ell \text{ via}$$

- $W_w(ng) = e^{2\pi i \langle w, \bar{n} \rangle} W_w(g) \quad \forall n \in N(\mathbb{R})$
- $W_w(gk) = k^{-1} \cdot W_w(g) \quad \forall k \in K$

<u>THM</u>: Suppose $w \neq 0$, ~~Then~~ F is a GWF of type (w, ℓ).

1) If $w \not\geq 0$, then $F \equiv 0$.
2) If $w \geq 0$, then $F(g) = \lambda W_w(g)$ for some $\lambda \in \mathbb{C}$.

<u>Consequently</u>: If φ is a MF on G_2 of wt ℓ, $\exists\, a_\varphi(w) \in \mathbb{C}$ s.t.

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \geq 0,\ w \text{ integral}} a_\varphi(w) W_w(g)$$

Moreover:

φ_N can be described explicitly in terms of hol MF of wt 3ℓ on GL_2

Def$^{\underline{n}}$: The $a_\varphi(w)$ are, by Def$^{\underline{n}}$, the F.C.s of φ.

Rmk: Gan-Gross-Savin, using a mult 1 result of N. Wallach, had previously defined the F.C.'s w/o using the explicit fcns $W_w(s)$ if

- $\mathrm{disc}(f_w) \neq 0$
- $\ell \geq 4$.