Modular forms on exceptional gps

1) What is G_2 and MFs on G_2?
2) Fourier expansion of MFs on G_2
3) Given examples & Thms
4) Beyond G_2

- $f : \mathfrak{h} \to \mathbb{C}$ be a wt $\ell > 0$, level Γ mod form

Recall: $\phi_f : SL_2(\mathbb{R}) \to \mathbb{C}$ as

- $\phi_f(g) = j(g, i)^{-\ell} f(g \cdot i)$
- $j(g, z) = cz + d$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then

(0) ϕ_f is of moderate growth

(1) $\phi_f(\gamma g) = \phi_f(g) \quad \forall \gamma \in \Gamma \leq SL_2(\mathbb{Z})$

(2) $k_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \underline{SO(2)}$

$$\phi_f(g k_\theta) = e^{-i\ell\theta} \phi_f(g)$$

(3) $D_{CR}\phi_f \equiv 0$ where:

$$sl_2(\mathbb{R}) \otimes \mathbb{C} = \underbrace{k_0 \otimes \mathbb{C}}_{} + \underbrace{p_0 \otimes \mathbb{C}}_{}$$

$$M_2(\mathbb{C})^{tr=0} = \text{Anti} + \text{Sym}$$

$$p_0 \otimes \mathbb{C} = \mathbb{C}X_+ + \mathbb{C}X_-$$

$$X_\pm = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}$$

$$D_{CR}\phi_f = X_-\phi_f = 0.$$

<u>Conversely</u> Suppose $\phi : SL_2(\mathbb{R}) \to \mathbb{C}$ satisfies (0) - (3).

Then

$f(z) = j(g_z, i)^{\ell} \phi(g_z)$ where

- $g_z \cdot i = z$

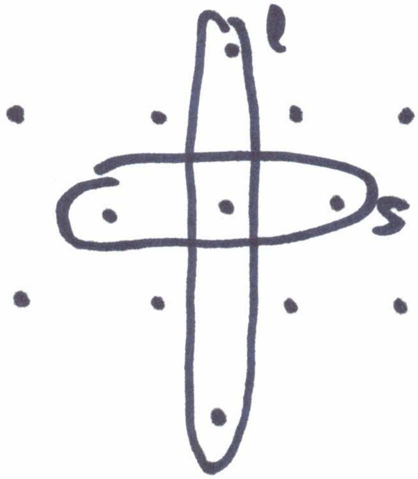
is well-defined, holom, wt ℓ, level Γ Mod form

Mod forms on G_2

- G_2: a simple noncompact Lie gp of dim$^{\underline{n}}$ 14

$\cup$

$K = (SU_{\ell}(2) \times SU_s(2)) / \{\pm 1\}$



$K \leq \mathbb{V}_\ell := \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \boxtimes \mathbb{1}$

__Note:__ The diagonal ± 1 acts trivially on $\mathbb{V}_\ell$

__Def^n__ (Gross-Wallach, Gan-Gross-Savin)

- Suppose $\Gamma \leq G_2$ is a congruence subgp

$$\left(\Gamma = G_2(\mathbb{Q}) \cap \underset{\substack{\cap \\ G_2(\mathbb{A}_f)}}{K_f} \right)$$

- $\ell > 0$ integer

A mod form on G_2 of wt ℓ & level Γ

is $\quad \phi : G_2 \longrightarrow \mathbb{V}_\ell$

st

0. ϕ has mod. growth
1. $\phi(\gamma g) = \phi(g) \ \forall \gamma \in \Gamma$
2. $\phi(gk) = k^{-1} \cdot \phi(g) \ \forall k \in K$
3. $D_\ell \varphi \equiv 0$

<u>OR:</u> $\varphi : G_2(\mathbb{Q}) \backslash G_2(\mathbb{A}) \to \mathbb{V}_\ell$ s.t. ...

<u>TO DO:</u>
1. What is G_2
2. What is D_ℓ
3. Examples & Thms about MFs on G_2

<u>UPSHOT:</u> MFs on G_2 have a classical F.E. & F.C.'s. The F.C.'s appear to be very arithmetic.

<u>What is G_2:</u> Will define a Lie alg / $\mathbb{Q}$

$$\mathfrak{g}_2 = \underbrace{\left(\mathfrak{sl}_3 = M_3^{\mathrm{tr}=0}\right)}_{\deg 0} + \underbrace{V_3(\mathbb{Q})}_{\deg 1} + \underbrace{V_3^\vee(\mathbb{Q})}_{\deg 2}$$

This is a $\mathbb{Z}/3$-grading: Meaning: if $X \in \deg i$
$Y \in \deg j$ then $[X, Y] \in \deg i+j$

<u>Here:</u> V_3 is 3-dim'l std rep of $\mathfrak{sl}_3$

$V_3^\vee$ is its dual

<u>A bracket:</u>

Suppose $\phi, \phi' \in \mathfrak{sl}_3$, $v, v' \in V_3$, $\delta, \delta' \in V_3^\vee$

- $[\phi, \phi'] = \phi \circ \phi' - \phi' \circ \phi$
- $[\phi, v] = \phi(v)$
- $[\phi, \delta] = \phi(\delta)$

Observe: $\Lambda^3 V_3 \simeq \mathbb{1}$

$\Rightarrow$ · $\Lambda^2 V_3 \simeq V_3^\vee$

· $\Lambda^2 V_3^\vee \simeq V_3$

Explicitly: $V_3 = \langle v_1, v_2, v_3 \rangle$ fixed basis

$V_3^\vee = \langle \delta_1, \delta_2, \delta_3 \rangle$ dual basis

· $v_i \wedge v_{i+1} = \delta_{i-1}$; $\delta_i \wedge \delta_{i+1} = v_{i-1}$

· $[v, v'] = 2 v \wedge v' \in \Lambda^2 V_3 \simeq V_3^\vee$

· $[\delta, \delta'] = 2 \delta \wedge \delta' \in \Lambda^2 V_3^\vee \simeq V_3$

· $[\delta, v] = 3 \underset{\underset{V_3 \otimes V_3^\vee = \mathrm{End}(V_3)}{\cap}}{v \otimes \delta} - \delta(v) \mathbb{1} \in \mathfrak{sl}_3$

· Everything else determined by antisymmetry & linearity

<u>Prop^{n}</u> $\mathfrak{g}_2$ is a simple Lie alg, i.e. the Jacobi identity is satisfied & there are no nontrivial ideals

$$Aut(\mathfrak{g}_2) = \{ g \in GL(\mathfrak{g}_2) : [gX, gY] = g[X,Y] \ \forall\, X, Y \in \mathfrak{g}_2 \}.$$

$$G_2 = Aut^{0}(\mathfrak{g}_2).$$

- Analogous procedure to define all exceptional gps

The root diagram of $\mathfrak{g}_2$

- Let $\mathfrak{h} \subseteq \mathfrak{sl}_3$ be the diagonal elts

$$= \{ q_1 E_{11} + q_2 E_{22} + q_3 E_{33} : q_1 + q_2 + q_3 = 0 \}$$

- Let $r_1, r_2, r_3 : \mathfrak{h} \to \mathbb{Q}$ be

$$r_j (q_1 E_{11} + q_2 E_{22} + q_3 E_{33}) = q_j .$$

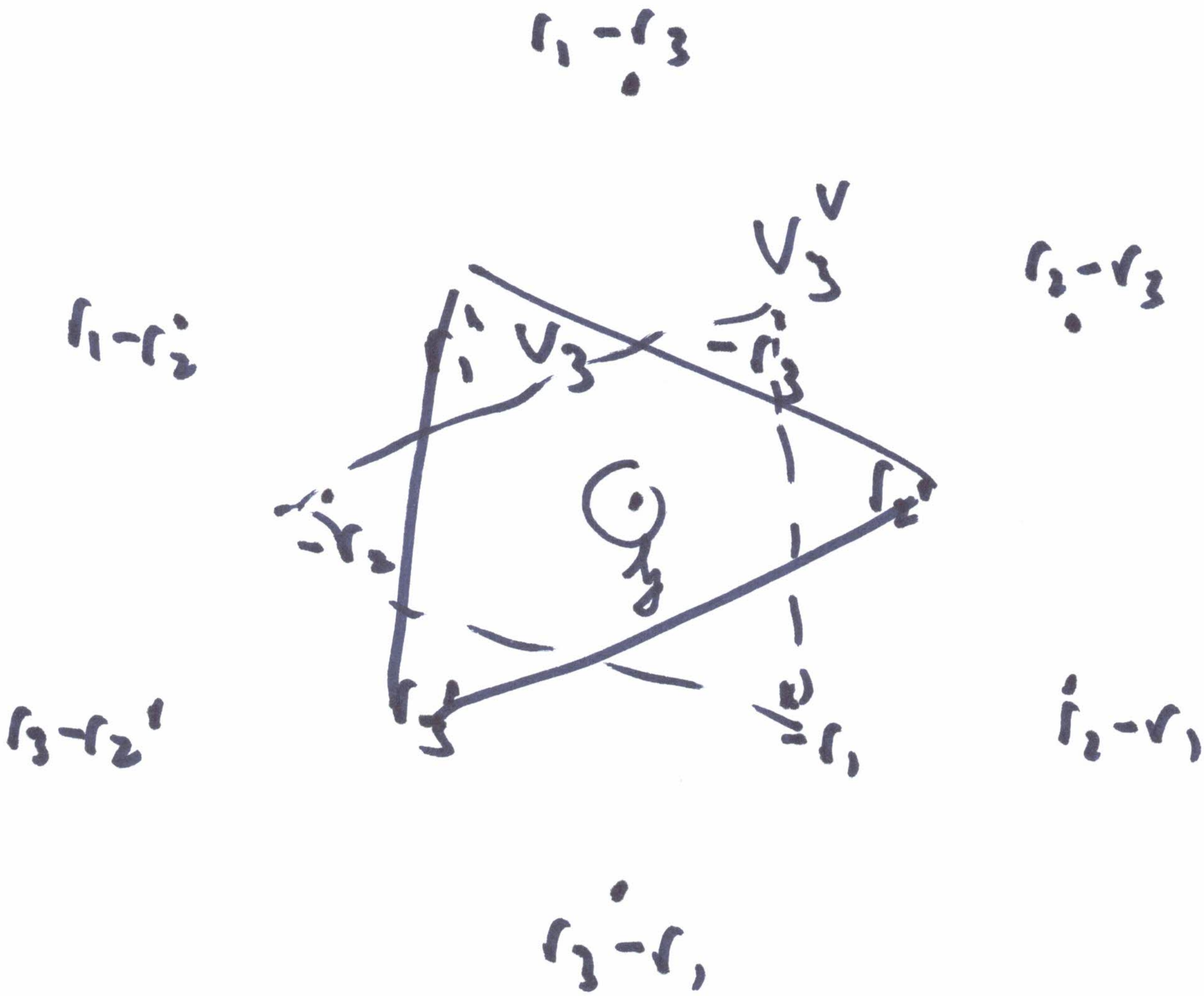
<u>note</u>: $r_1 + r_2 + r_3 = 0.$

<u>wts of $\mathfrak{h}$ on $\mathfrak{g}_2$</u>:

$$\mathfrak{g}_2 = \mathfrak{sl}_3 + V_3 + V_3^\vee$$

on V_3: r_1, r_2, r_3 on $\mathfrak{sl}_3$: $\{ r_i - r_j \}_{i \neq j}$

on $V_3^\vee$: $-r_1, -r_2, -r_3$


r1 - r3
V3v
r2 - r3
r1 - r2
V3
r1
-r3
r2
-r2
r3
-r1
r3 - r2
r2 - r1
r3 - r1

The diff op

Define: $\Theta : \mathfrak{g}_2 \otimes \mathbb{R} \to \mathfrak{g}_2 \otimes \mathbb{R}$, a Cartan involution

Explicitly:

$$\Theta : \mathfrak{sl}_3 \longrightarrow \mathfrak{sl}_3 \quad \text{as} \quad X \longmapsto -{}^t X$$

$$V_3 \longleftrightarrow V_3^\vee$$

$$v_j \longleftrightarrow \delta_j$$

$$\mathfrak{k}_0 = \mathfrak{g}_2 \otimes \mathbb{R}^{\Theta = 1}, \quad \mathfrak{p}_0 := (\mathfrak{g}_2 \otimes \mathbb{R})^{\Theta = -1}$$

$$K = \{ g \in G_2 : \mathrm{Ad}(g) \circ \Theta = \Theta \circ \mathrm{Ad}(g) \}$$

$$\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbb{C} : \quad \mathfrak{sl}_2 + \mathfrak{sl}_2$$

$$\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C} : \quad V_2 \otimes \mathrm{Sym}^3(V_2)$$

Will have: $D_\ell = \underline{\mathrm{pr}} \circ \tilde{D}_\ell$

where:

- Suppose $\varphi : G_2 \to V_\ell = \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \boxtimes \mathbb{1}$ satisfies $\varphi(gk) = k^{-1} \cdot \varphi(g)\ \forall k \in K$

- Let $\{X_\alpha\}_\alpha$ be a basis of $\mathfrak{p}$

  $\{X_\alpha^\vee\}_\alpha$ the dual basis of $\mathfrak{p}^\vee$

Then $\tilde{D}_\ell \varphi = \sum_\alpha X_\alpha \varphi \otimes X_\alpha^\vee \in V_\ell \otimes \mathfrak{p}^\vee$

where: $X_\alpha \varphi$ is the diff of right reg action

I.e. if $X \in \mathfrak{p}_0$ then

$$(X\varphi)(g) = \frac{d}{dt}\left(\varphi(g e^{tX})\right)\Big|_{t=0}.$$

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$$\mathbb{V}_\ell \otimes \rho^\vee = \left(S^{2\ell} \boxtimes \mathbb{1}\right) \otimes \left(V_2 \boxtimes \mathrm{Sym}^3(V_2)\right)$$

$$= \left(S^{2\ell+1} + S^{2\ell-1}\right) \boxtimes S^3(V_2)$$

$$\xrightarrow{\;pr\;} S^{2\ell-1}(V_2) \boxtimes S^3(V_2)$$

$$\underline{\underline{D_\ell}} = pr \circ \tilde{D}_\ell .$$

<u>Rmk:</u> $G_2 \rightsquigarrow Sl_2$

$D_\ell \rightsquigarrow D_{CR}$
