Addendum

- Local case: should also consider

$$E = F \times F \to V = V_0 \times V_0^\vee$$

$$U(V) \simeq GL(V_0)$$

Weil Rep: $GL(V_0) \times GL(W_0)$

$$S(V_0 \otimes W_0)$$

(Minguez)

- Why just symplectic group?

$$\mathrm{Aut}(\mathbb{O}) \times \mathrm{Aut}(J)$$

$$G_2 \times \left\{ \begin{array}{lll} F_4 & \text{in} & E_8 \\ PGSp_6 & & E_7 \\ PGL_3 & & E_6 \\ PU_3^K & & E_6^K \\ SO_3 & & F_4 \end{array} \right.$$

Project Group

GLOBAL / k

$V = \langle 1 \rangle$, $W_r^{\varepsilon} = W_0^{\varepsilon} \oplus \mathbb{H}^r$

dim $= 2r+1$

$\chi \leq \mathcal{A}(U(V))$

Know : $\Theta_r^{\varepsilon}(\chi) \neq 0 \quad \forall r > 0$

$\cap$ $\mathcal{A}_2(U(W_r^{\varepsilon}))$ (Stable range)

$r = 0$?

THM: $\Theta_{W_0^{\varepsilon}}(\chi) \neq 0$

$\Longleftrightarrow$ (a) $\forall v,\quad \Theta_{W_0^{\varepsilon_v}}(\chi_v) \neq 0$

(controlled by a local root no. condition)

(b) $L(\frac{1}{2}, \chi_E \cdot \chi_W^{-1}) \neq 0$

Note : $\varepsilon_v = \varepsilon(\frac{1}{2}, \chi_{E,v}\chi_{W,v}^{-1}, \psi(\mathrm{Tr}(\delta -)))$

$+1 = \prod_v \varepsilon_v = \varepsilon(\frac{1}{2}, \chi_E \chi_W^{-1})$

<u>Proof</u> (Sketch)

For $\phi \in \omega_\psi \rightsquigarrow \Theta(\phi)$

$$\Theta(\phi, \chi)(g) = \int_{[U(V)]} \phi(g, h)\, \chi(h)^{-1}\, dh$$

$\nearrow$ $A(U(W_0))$

Is this function nonzero for some ϕ?

Compute inner product of $\Theta(\phi, \chi)$:

$$\langle \Theta(\phi, \chi), \Theta(\phi, \chi) \rangle \qquad \Theta(\phi,\chi)\,\overline{\Theta(\phi,\chi)}$$

$\Theta(1)$ " Eis. series

$$\begin{tikzcd}
U(V^\square) \arrow[dr, dash] & U(W_0) \times U(W_0) \arrow[dl, dash] \\
U(V) \times U(V^-) \arrow[u] & U(W_0)^\Delta \arrow[u, "\Delta"']
\end{tikzcd}$$

$\chi \quad \chi^{-1} \qquad\qquad 1$

$$\langle \theta(\phi,\chi), \theta(\phi,\chi) \rangle$$

Doubling see-saw
+ Siegel-Weil

$$Z(0,\phi,\chi)$$ Doubling Zeta integral

Apply Ellen.

$$(\ast)\cdot L(\tfrac{1}{2}, \chi_E \vec{\chi}_W)$$

(Rallis inner product formula)

Howe - PS

$V = \langle 1 \rangle$, $W = W_0 \oplus \mathbb{H}$

Consider $\Theta_W(1) \subseteq A_2(U(W))$

$0 \neq$ irred.

Know: for $\neq$ a.a. v,

$\Theta_W(1)_v \longleftrightarrow \mathrm{Ind}_{B_v}^{U(W_v)} |\;|_v^{-\frac{1}{2}} \otimes 1_v$

$\searrow$ non-tempered

Cuspidality ?

Yes ? QED

No ? This means $\Theta_{W_0}(1) \neq 0$

Do above with $W' = W_0' \oplus \mathbb{H}$

Pick 2 places v_1, v_2 of k & replace W_{0,v_1}, W_{0,v_2} by their cousins $\longrightarrow W_0'$

Arthur's Conjecture

GOAL: Classify constituents of

$$A_2(G) \circlearrowright G(\mathbb{A})$$

(Basic Hypothesis): $\exists\ L_F \ (\approx \mathrm{Gal}\,\bar{F}/F)$

s.t. $\left\{\begin{matrix}\text{Irred. } n\text{-dim} \\ \text{reps of } L_F\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Cuspidal} \\ \text{rep of} \\ GL(n)\end{matrix}\right\}$

& st for all v, have

$$\begin{array}{ccc} \text{Weil-Deligne group} \leftarrow L_{F_v} & \longrightarrow & L_F \\ \wr\wr & & \\ (\mathrm{Gal}\,\bar{F}_v/F_v) & & \end{array}$$

Defn: $\pi = \bigotimes_v \pi_v$ & $\pi' = \bigotimes_v \pi'_v$

are nearly equiv if

for a.a. v, $\pi_v \simeq \pi'_v$

(A) $A_2(G) = \bigoplus_{\psi} A_{\psi}$

near eq. classes

where

$\psi : L_F \times SL_2 \longrightarrow {}^L G = G^\vee \rtimes \mathrm{Gal}(\bar{F}/F)$

Γ_F

s.t. · $\psi(L_F)$ is bounded

· $Z_{G^\vee}(\psi)^{\Gamma_F} / Z_{G^\vee}$ is finite

(elliptic A-parameters)

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Given ψ, need to def. A_ψ.

$\psi \rightsquigarrow$ • (Global Component/Centralizer)

$$S_\psi = Z_{G^\vee}(\psi) / Z(G^\vee)^{\Gamma_F}$$

(finite)

• $\psi_v : L_{F_v} \times SL_2 \hookrightarrow L_F \times SL_2$

$$\begin{tikzcd} L_{F_v} \times SL_2 \arrow[r, hook] \arrow[drr, dashed] & L_F \times SL_2 \arrow[dr, "\psi"] & \\ & & {}^L G \end{tikzcd}$$

• (Local comp. gp)

$$S_{\psi_v} = \pi_0\left(Z_{G^\vee}(\psi_v) / Z(G^\vee)^{\Gamma_{F_v}} \right)$$

(finite

• $S_\psi \xrightarrow{\Delta} \prod_v S_{\psi_v} =: S_{\psi, \mathbb{A}}$ (Compact)

• $\varepsilon_\psi : S_\psi \longrightarrow \langle \pm 1 \rangle$

quad. char.

(B) For all v, have a finite set

$$\Pi_{\psi_v} = \{ \pi_{\eta_v} : \eta_v \in \mathrm{Irr}\, S_{\psi_v} \}$$

of unitary reps of $G(F_v)$

s.t. for a.a. v.

π_{1v} is irred. unram. with

Satake parameter

$$\psi_v \left(\mathrm{Frob}_v , \begin{pmatrix} q_v^{\frac{1}{2}} & \\ & q_v^{-\frac{1}{2}} \end{pmatrix} \right) \in {}^L G$$

Key Point : If $\psi(SL_2) = 1$,

then π_{1v} is tempered.

If not, π_{1v} is non-temp

Set

$$\Pi_\psi = \bigotimes_v \Pi_{\psi v}$$

$$= \left\{ \bigotimes_v{}' \pi_{\psi v} : \pi_{\psi v} \in \Pi_{\psi v} \right\}$$

$$\parallel$$

$$\pi_\eta \,.\; \eta \in \mathrm{Irr}\, S_{\psi,\mathbb{A}}$$

$$\boxed{\begin{gathered} A_\psi = \bigoplus_{\eta \in \mathrm{Irr}\, S_{\psi,\mathbb{A}}} m_\eta \pi_\eta \\ \text{with} \\ m_\eta = \dim \mathrm{Hom}_{S_\psi}(\epsilon_\psi, \eta) \end{gathered}}$$

Def of ε_ψ

$$\begin{tikzcd} (L_F \times SL_2) \times S_\psi \arrow[r, "\psi \times id"] & {}^L G / Z(G^\vee)^{\Gamma_F} \arrow[d, "Ad"] \\ & \mathrm{Lie}(G^\vee) \end{tikzcd}$$

$$\mathrm{Lie}(G^\vee) = \mathfrak{g}^\vee$$

$$\bigoplus_{i \in I} \rho_i \otimes S_{r_i} \otimes \eta_i = \mathfrak{g}^\vee$$

$(S_r = r\text{-dim irrep of } SL_2)$

Set $T = \left\{ i \in I \;\middle|\; \begin{array}{l} r_i \text{ even} \\ \eta_i \text{ orthog.} \\ \rho_i \text{ symp.} \end{array} \right\}$

$\varepsilon(\tfrac{1}{2}, \rho_i) = -1 \longleftarrow \rho_i$ symp.

Def: $\varepsilon_\psi : S_\psi \longrightarrow \langle \pm 1 \rangle$

$$\varepsilon_\psi(s) = \prod_{i \in T} \eta_i(s)$$

Eg: $\psi(SL_2) = \{1\}$, $T = \emptyset$

$\varepsilon_\psi = 1$

$G = \underline{U(n)}$ E/F

$G^\vee = GL_n(\mathbb{C}) \triangleleft {}^LG = GL_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F)$

$\psi : L_F \times SL_2 \longrightarrow {}^LG$

$\cup$

$\psi|_E : L_E \times SL_2 \longrightarrow G^\vee = GL_n(\mathbb{C})$

conjugate self-dual rep,

of sign $(-1)^{n-1}$

~~Eg~~

Eg: $U(3)$

$\psi|_{L_E} : L_E \times SL_2 \longrightarrow GL_3(\mathbb{C})$

$\psi|_{L_E} = \mu \oplus \chi \otimes S_2$

μ, χ chars ~~of L_E~~ of auto chars of $\mathbb{A}_E^\times / E^\times$

$\cancel{\mu : \mathbb{A}_E^\times / \mathbb{A}}$ $\quad \mu|_{\mathbb{A}_F^\times} = 1$

s.t. $\quad \chi|_{\mathbb{A}_F^\times} = \omega_{E/F}$

$$S_\psi = \mu_2 \xrightarrow{\Delta} \prod_v S_{\psi_v}$$

$$S_{\psi_v} = \begin{cases} \mu_2 & \text{if } v \text{ inert in } E \\ 1 & \text{if not} \end{cases}$$

$$\varepsilon_\psi : \mu_2 \longrightarrow \langle \pm 1 \rangle$$

$$\varepsilon_\psi = \begin{cases} \text{triv.} & \text{if } \varepsilon(\tfrac{1}{2}, \chi\mu^{-1}) = 1 \\ \text{non-triv.} & \text{if } \text{———} = -1 \end{cases}$$

$$\Pi_{\psi_v} = \begin{cases} \{\pi_v^+, \pi_v^-\} & \text{if } v \text{ inert} \\ \{\pi_v^+\} & \text{if not} \end{cases}$$

$$m(\pi^\varepsilon) = \begin{cases} 1 & \text{if } \prod_v \varepsilon_v = \varepsilon(\tfrac{1}{2}, \chi\mu^{-1}) \\ 0 & \text{if not} \end{cases}$$

For a.a. v.

$$\pi_{1_v} = \pi_v^{+}$$

$$\hookrightarrow \mathrm{Ind}_{B_v}^{U_3} \chi|\cdot|_v^{-\frac{1}{2}} \otimes \tilde{\mu}$$

This is the Howe–PS eg.