More on Aut. Forms + approach to studying $L(s,\pi)$

π cusp aut repn of unitary gp

Recall: saw several perspectives on aut. forms on unitary gps:

- As fcns on generalization of upper half plane
- As fcns of $G(\mathbb{R})$, $G(\mathbb{A})$

$\boxed{\text{G denotes a unitary gp}}$

Now, aut. forms on unitary gps as sections of line (or vector) bundle over moduli space

M. Forms:

$$\begin{array}{l} \mathcal{E} = \text{univ elliptic curve} \qquad \Omega_{\mathcal{E}/\mathcal{M}} \\ \big\downarrow \pi \\ \mathcal{M} \end{array}$$

$\mathcal{M}$: modular curve (parametrizing Elliptic curves w/ level str)

$\underline{\omega} := \pi_* \Omega^1_{\mathcal{E}/\mathcal{M}}$

A <u>modular form</u> of wt k is an elt of $H^0(\mathcal{M}, \underline{\omega}^{\otimes k})$

Could think of m. form as rule F that maps pairs (E, ω) to elts of $\mathbb{C}$

$F(E,\omega) \in \mathbb{C}$

$\omega \in \Omega_{E/\mathbb{C}}$ s.t. $F(E, \lambda\omega) = \lambda^{-k} F(E,\omega) \quad \forall \lambda \in \mathbb{C}^\times$

or equiv

as rule $\tilde{F}$ that maps e.c. E to an elt $\omega \in \Omega_{E/\mathbb{C}}$

$(\tilde{F}(E) = F(E,\omega) \cdot \omega^{\otimes k})$

Connection with last time:

$(E,\omega) \longleftrightarrow L_{(E,\omega)} \longleftrightarrow \mathbb{Z} + \tau\mathbb{Z}$

$F(E,\omega) \longleftrightarrow f_F(\tau)$

Aut. Forms on unitary gps, similarly, arise as global sections of V. bundle over unitary Sh. variety $\mathcal{M}$.

parametrize A.V's with
- polarization
- endomorphism
- level structure

Rmk: The $\mathbb{C}$-pts of $\mathcal{M}$ can be identified with

$$\underbrace{G(\mathbb{Q})\backslash G(\mathbb{A})/K\cdot K_\infty}_{\parallel}$$

finite disjoint union of copies of symmetric space (e.g. $\mathfrak{h}_n$) for our unitary gp

Can view an aut form as fcn

F s.t.

$$F(\underline{A}, g\cdot \ell) = \rho({}^t g)^{-1} F(\underline{A}, \ell)$$

$\ell = (\ell_+, \ell_-)$

ordered basis for $\Omega_{A/\mathbb{C}}$

$g \in GL_a \times GL_b$ (sig (a,b))

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· Can also reformulate in terms of lattices and identify with an aut. form on symm space (e.g. $\mathfrak{h}_n$)

$$\begin{array}{c} \mathcal{A} \\ \downarrow \pi \\ \mathcal{M} \end{array} \qquad \underline{\omega} := \pi_* \Omega_{\mathcal{A}/\mathcal{M}}$$

From $\underline{\omega}$, can build a sheaf of " $\underline{\omega}^{\rho}$ " aut. forms.

GOAL FOR REST TODAY:

Introduce approach to studying certain L-fcns, with an emphasis on "doubling method"

Motivating Example:

$f(q) = \sum_{n\geq 1} a_n q^n$ wt k cusp form

$g(q) = \sum_{n\geq 0} b_n q^n$ wt l m. form

$a_n, b_n \in \overline{\mathbb{Q}}$

Rankin-Selberg product is

$$D(s,f,g) = \sum_{n\geq 1} \frac{a_n b_n}{n^s}$$

- Shimura pr'd

$$\frac{D(m,f,g)}{\langle f, f\rangle_{pet}} \in \pi^k \overline{\mathbb{Q}}$$

for $l<k$ and $\frac{k+l-2}{2} < m < k$ $(m\in\mathbb{Z})$

Pf relies on realization that

Eis. series

$$D(k-1-r, f, g) = c\pi^k \langle \tilde{f}, g\, \delta_\lambda^{(r)} E \rangle_{Pet}$$

$$\tilde{f}(q) = \sum_{n \geq 1} \overline{a_n}\, q^n$$

"Maass-Shimura differential operator"

wt λ: $k-1-2r$

$\langle f, g \rangle_{Pet}$ is the Petersson pairing, i.e.

$$(*) \int \overline{f}(z)\, g(z)\, y^{k-2}\, dx\, dy$$

Fund domain

$(\Gamma_1(N) \backslash \mathfrak{h})$

Recipe for proving alg of certain vals of L-fcns

① Find a Petersson-style pairing of aut. forms (int. against Ei. series E) that:
- factors into an Euler product
- has a fcn'l eqn
- can be mero. cont'd to $\mathbb{C}$

② Pv ~~the~~ "appropriate" rationality results for E

③ Express a familiar aut L-fcn in terms of this pairing

Rmk: These are hard steps and depending on your setting, might not have been done yet.

Setup:

$$\begin{array}{c} K \\ \big| \text{ quad. ext.} \\ \mathbb{Q} \end{array}$$

- V/K n-dim'l v.s. with nondeg herm pairing $\langle\, ,\, \rangle_V$
- $G := U(V, \langle\, ,\, \rangle_V)$
- $W = V \oplus V$ with herm. pairing

$$\langle (u, v), (u', v') \rangle := \langle u, u' \rangle_V - \langle v, v' \rangle_V$$

- $H := U(W, \langle\, ,\, \rangle_W)$
- $$\begin{array}{ccccc} G & \times & G & \hookrightarrow & H \\ \| & & \| & & \\ U(V, \langle\, ,\, \rangle) & & U(V, -\langle\, ,\, \rangle) & & \mathrm{sig}(\underbrace{a+b}_{n}, \underbrace{a+b}_{n}) \\ \mathrm{sig}(a,b) & & \mathrm{sig}(b,a) & & \end{array}$$

Next time:

Introduce doubling integral (after producing E. series)