

More on Aut. Forms
 + approach to studying

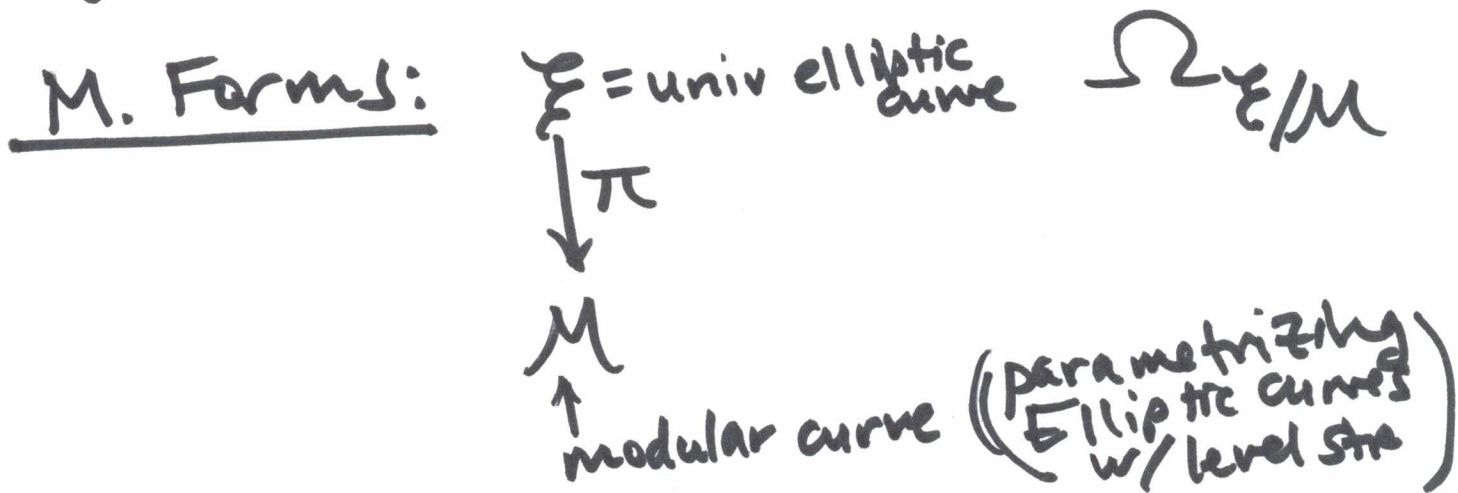
$L(s, \pi)$
 \uparrow
 π cusp aut repr of unitary gp

Recall: saw several perspectives on aut. forms on unitary gps:

- As fns on generalization of upper half plane
- As fns of $\begin{cases} G(\mathbb{R}) \\ G(\mathbb{A}) \end{cases}$

G denotes a unitary gp

Now, aut. forms on unitary gps as sections of line (or vector) bundle over moduli space



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$$\underline{\omega} := \pi_* \Omega^1_{E/M}$$

A modular form of wt k is an elt of $H^0(M, \underline{\omega}^{\otimes k})$

Could think of m. form as rule F that maps pairs

(E, ω) to elts of \mathbb{C}

$\omega \in \Omega_{E/\mathbb{C}}$ s.t. $F(E, \omega) \in \mathbb{C}$

$$F(E, \lambda \omega) = \lambda^{-k} F(E, \omega) \quad \forall \lambda \in \mathbb{C}^\times$$

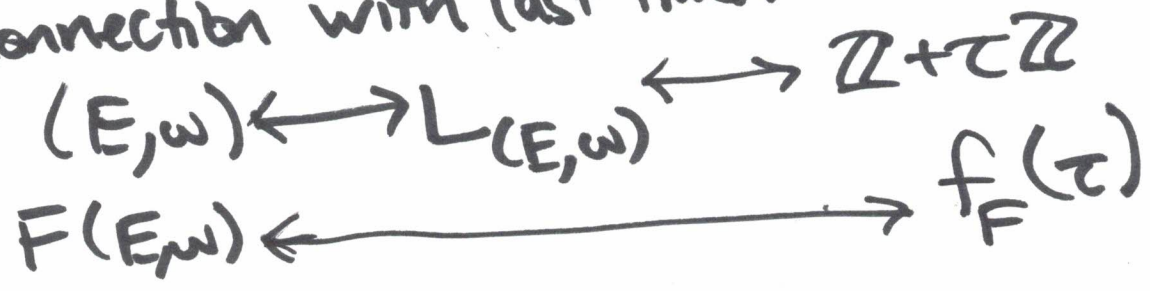
or equiv

as rule

\tilde{F} that maps e.c. E to an elt $\omega \in \Omega_{E/\mathbb{C}}$

$$(\tilde{F}(E) = F(E, \omega) \cdot \omega^{\otimes k})$$

Connection with last time:



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Aut. Forms on unitary gps, similarly,
arise as global sections of v. bundle
over unitary Sh. variety \mathcal{M} .

↑
parametrize A.V.'s with

- polarization
- endomorphism
- level structure

Rmk: The \mathbb{C} -pts of \mathcal{M} can be
identified with $\underbrace{G(\mathbb{A})/K \cdot K_\infty}_{G(\mathbb{Q})}$

||
finite disjoint union
of copies of symmetric
space (e.g. \mathfrak{h}_n) for
our unitary gp

Can view an aut form as fun

F ~~satisfies~~ s.t.
 $= (l_+, l_-)$

$$F(\underline{A}, \underset{\uparrow}{g \cdot l}) = \rho(\tau g)^{-1} F(\underline{A}, l)$$

ordered basis
for $\Omega_{\mathbb{A}/\mathbb{C}}$

$g \in GL_a \times GL_b$
(sig (a, b))

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Can also reformulate in terms of lattices and identify with an aut. form on symm space (e.g. h_n)



$$\underline{\omega} := \pi_* \Omega_{\mathcal{A}/\mathcal{M}}$$

From $\underline{\omega}$, can build a sheaf of aut. forms.
 " ω^p "

5 GOAL FOR Rest TODAY:

Introduce approach to studying certain L-funs, with an emphasis on "doubling method"

Motivating Example:

$f(q) = \sum_{n \geq 1} a_n q^n$ wt k cusp form

$g(q) = \sum_{n \geq 0} b_n q^n$ wt l m. form
 $a_n, b_n \in \overline{\mathbb{Q}}$

Rankin-Selberg product is

$D(\overset{S, f, g}{f, g}) = \sum_{n \geq 1} \frac{a_n b_n}{n^s}$

Shimura pr'd

$\frac{D(m, f, g)}{\langle f, f \rangle_{\text{pet}}} \in \pi^k \overline{\mathbb{Q}}$

for $l < k$ and $\frac{k+l-2}{2} < m < k$ ($m \in \mathbb{Z}$)

6 Pf relies on realization that

$D(k-1-r, f, g) = c\pi^k \langle \tilde{f}, g \rangle_{\text{Pet}}$

$\tilde{f}(q) = \sum_{n \geq 1} \overline{a_n} q^n$

$\langle \tilde{f}, g \rangle_{\text{Pet}} = \int_{\Sigma} \tilde{f} \circ \sigma_{\lambda}^{(r)} g$

↑ "Maass-Shimura differential operator"

wt $\lambda: k-1-2r$

↑ E

↑ EIS. series

$\langle f, g \rangle_{\text{Pet}}$ is the Petersson pairing, i.e.

$$(*) \int_{\text{Fund domain}} \overline{f(z)} g(z) y^{k-2} dx dy$$

($\Gamma \backslash \mathbb{H}$)

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Recipe for proving alg of certain vals of L-fun

- ① Find a Petersson-style pairing of aut. forms (int. against E -form) that:
 - factors into an Euler product
 - has a fun'l eqn
 - can be zero. cont'd to \mathbb{C}
- ② PV ~~the~~ "appropriate" rationality results for E
- ③ Express a familiar aut L-fun in terms of this pairing

Rmk: These are hard steps and depending on your setting, might not have been done yet.

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Setup:
 K
 quad. m.
 \mathbb{Q}

V/K n -dim'd v.s. with
 nondeg herm pairing
 \langle, \rangle

$G = U(V, \langle, \rangle)$

$W = V \oplus V$ with herm. pairing

$$\langle (u, v), (u', v') \rangle = \langle u, u' \rangle_V - \langle v, v' \rangle_V$$

$H = U(W, \langle, \rangle_W)$

$G \times G \longrightarrow H$
 $\begin{matrix} \text{sig}(a, b) & \text{sig}(b, a) & \text{sig}(\underbrace{a+b}_n, \underbrace{a+b}_n) \end{matrix}$
 $U(V, \langle, \rangle) \quad U(V, -\langle, \rangle) \quad \text{sig}(a+b, a+b)$

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Next time:

Introduce doubling integral
(after producing E. series)