Lecture 4

The j-invariant

In the last lecture, we saw that the j-invariant of a lattice Λ is the same as the j-invariant of the associated elliptic curve E_Λ. We also noted that the surjectivity is important in showing that an elliptic curve can be associated to a lattice. In this section, we will briefly recap the definition of the j-invariant. We'll also follow Sutherland's approach for showing that the j-invariant gives a surjection $\mathcal{H} \to \mathbb{C}$ which then gives a surjection from $\mathcal{F}$ to $\mathbb{C}$.

Recall that for a lattice Λ, the Eisenstein series of weight $2k$ for Λ is the series

$$G_{2k}(\Lambda) = \sum_{\lambda \in \Lambda}{}' \lambda^{-2k}.$$

Recall also that for each Λ, we defined quantities $g_2(\Lambda)$, $g_3(\Lambda)$, and $j(\Lambda)$ given by

$$g_2(\Lambda) = 60G_4(\Lambda), \;\; g_3(\Lambda) = 140G_6(\Lambda)), \;\; \text{and } j(\Lambda) = 1728\frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27g_3(\Lambda)^2}.$$

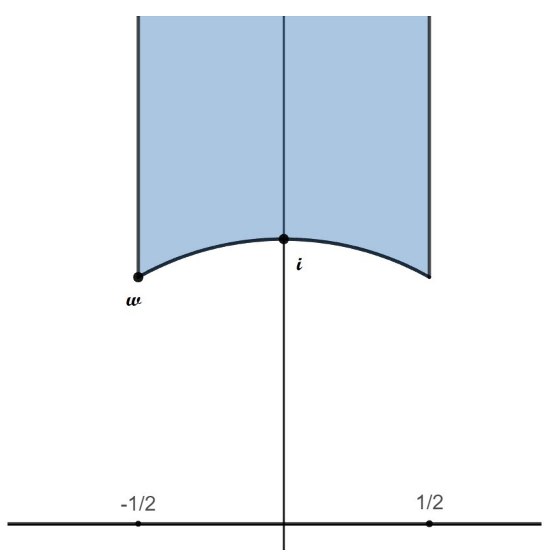
Letting $j(\tau) \coloneqq j(\Lambda_\tau)$ gives us a function $j : \mathcal{H} \to \mathbb{C}$.

On the one hand, as stated in the last lecture, $g_2(\Lambda)^3 - 27g_3(\Lambda)^2$ is never 0. On the other hand, one can show ([2, I.3.4.2]) that for $k \geq 2$, $\lim_{\tau\to\infty} G_{2k}(\tau) = 2\zeta(2k)$, where $\zeta(z)$ denotes the Riemann zeta function. From this, we have $\lim_{\tau\to\infty} g_2(\Lambda)^3 - 27g_3(\Lambda)^2 = 0$, so that j is unbounded, and therefore nonconstant.

By the Open Mapping Theorem, since $\mathcal{H}$ is an open subset of $\mathbb{C}$ and j is holomorphic on $\mathcal{H}$, its image $j(\mathcal{H})$ is open. The image is also closed ([4, Thm 16.11]); since $\mathbb{C}$ is connected, the only nonempty subset of $\mathbb{C}$ which is both open and closed is $\mathbb{C}$ itself, hence $j(\mathcal{H}) = \mathbb{C}$.

Now, for every $\tau \in \mathcal{H}$, $\Lambda_\tau = \Lambda_{\tau+1}$, so that $j(\tau) = j(\tau+1) = j(T\tau)$. In addition, Λ_1 and Λ_2 are homothetic if and only if $j(\Lambda_1) = j(\Lambda_2)$. Thus, for every $\tau \in \mathcal{H}$, since $(-1/\tau)\Lambda_\tau = \mathbb{Z} + (-1/\tau)\mathbb{Z} = \Lambda_{-1/\tau}$, we have Λ_τ and $\Lambda_{-1/\tau}$ are homothetic, so $j(\tau) = j(-1/\tau) = j(S\tau)$. Putting these together, we see that j yields a well-defined, surjective function $Y(1) \to \mathbb{C}$.

In lecture 1, we showed that the set $\mathcal{F}$ (pictured below) is a fundamental domain for the $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathcal{H}$.



If we imagine gluing the vertical portions of the boundary together (identifying τ and $\tau \pm 1$) and gluing together the portion of the boundary along the unit circle (identifying τ and $-1/\tau$), the resulting space is homeomorphic to a 2-sphere with one point missing.

Definition of a Riemann surface

Our goal for this lecture is to show that the space that we obtain from compactifying the set $Y(1)$ is a compact Riemann surface. The idea behind a Riemann surface is that it should look like $\mathbb{C}$ locally. Not only can we make sense of holomorphic maps on such surfaces so that we can study them as analytic spaces, when they are compact we can also view them as algebraic objects, since every compact Riemann surface is a projective variety. These are important and useful properties of compact Riemann surfaces, but what does it mean to "make sense of holomorphic maps"? To work with holomorphic maps on a Riemann surface X, we need some way of identifying open subsets of the surface with open subsets of $\mathbb{C}$ (where "holomorphic map" has a less seemingly-ambiguous meaning). That suggests one part of the definition: we want some way of mapping open sets of X to open subsets of $\mathbb{C}$. We may have more than one way of mapping an open set (in particular an intersection of two open sets) of X to $\mathbb{C}$; we want there to be some compatibility in these mappings. In other words, we would like X to have a complex structure.

Definition 1 *If X is a topological space, a complex structure on X is an open cover $\{V_\alpha\}$ of X together with homeomorphisms*

$$\psi_\alpha : V_\alpha \to U_\alpha,$$

such that U_α is an open subset of $\mathbb{C}$ and such that for all α, β with $V_\alpha \cap V_\beta \neq \emptyset$, the map

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(V_\alpha \cap V_\beta) \to \psi_\beta(V_\alpha \cap V_\beta)$$

is holomorphic.

Now that we have the definition of a complex structure, we can state the definition of a Riemann surface.

Definition 2 *A Riemann surface is a connected Hausdorff space with a complex structure.*

Thus, our goal for this lecture is to show that $X(1)$, the space obtained by compactifying $Y(1)$ is a compact, connected, Hausdorff space with a complex structure.

The quotient topology on $Y(1)$

To begin with, we will discuss the topology on $Y(1)$. We saw in lecture 1 that we cannot take for granted that the nice topological properties of $\mathcal{H}$ (e.g., the fact that $\mathcal{H}$ is Hausdorff) will be inherited by $Y(1)$. Let $\pi : \mathcal{H} \to Y(1)$ be the map $\pi(\tau) = \mathrm{SL}_2(\mathbb{Z})\tau$. The quotient topology on $Y(1)$ is given by

$$V \subseteq Y(1) \text{ is open if and only if } \pi^{-1}(V) \text{ is open.}$$

This definition immediately shows that π is continuous. Therefore, since $\mathcal{H}$ is connected, $Y(1)$ is connected. Next suppose $U \subseteq \mathcal{H}$ is open. Observe that:

(i) For each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\gamma : \mathcal{H} \to \mathcal{H}$, $\tau \mapsto \gamma\tau$ is a homeomorphism and therefore γU is open for each open $U \subseteq \mathcal{H}$.

(ii) We can write $\pi^{-1}(\pi(U)) = \bigcup_{\gamma\in \mathrm{SL}_2(\mathbb{Z})} \gamma U$.

Thus $\pi(U)$ is open, so π is an open map. Using this and the following lemma, we will show that $Y(1)$ is Hausdorff.

Lemma 3 *For any $\tau_1, \tau_2 \in \mathcal{H}$, there exist neighborhoods U_1, U_2 of τ_1, τ_2 such that*

$$\gamma U_1 \cap U_2 \neq \emptyset \iff \gamma\tau_1 = \tau_2$$

Remark 4 *The proof below follows [4, Lemma 19.1, 19.2].*

Proof: Let W_1, W_2 be open neighborhoods of τ_1, τ_2 with compact closures $K_1, K_2 \subset \mathcal{H}$ (respectively). We begin by showing that the set $\mathcal{I} = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma K_1 \cap K_2 \neq \emptyset\}$ is finite. Suppose $\alpha \in \gamma K_1 \cap K_2$ so that $\alpha = \gamma\beta$ for some $\beta \in K_1$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\mathrm{Im}(\alpha) = \mathrm{Im}(\gamma\beta) = \frac{\mathrm{Im}(\beta)}{|c\beta + d|^2},$$

so $|c\beta + d|^2 = \frac{\mathrm{Im}(\beta)}{\mathrm{Im}(\alpha)}$. Since K_1 and K_2 are compact, $\frac{\mathrm{Im}(\beta)}{\mathrm{Im}(\alpha)}$ achieves some maximum, so that as $|c\beta + d|^2$ is bounded. This implies there are finitely many pairs $c, d \in \mathbb{Z}$ such that (c, d) is the bottom row of γ for some $\gamma \in \mathcal{I}$. Fixing such a $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, if $\alpha = \tau\beta$ for $\alpha \in K_2$, $\beta \in K_1$, then

$$|\alpha| = |\gamma\beta| \implies |a\beta + b| = |\alpha||c\beta + d|.$$

As $|c|, |d|, |\alpha|$, and $|\beta|$ are each bounded, this implies $|a\beta + b|$ is bounded as well, so there can be only finitely many $a, b \in \mathbb{Z}$ for which $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{I}$.

Next, consider $\mathcal{I}' = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma K_1 \cap K_2 \neq \emptyset, \gamma\tau_1 \neq \tau_2\}$. Since $\mathcal{I}' \subseteq \mathcal{I}$, this set is finite. If $\mathcal{I}' \neq \emptyset$, for each, $\gamma \in \mathcal{I}'$, let $U_{1,\gamma}$ be a neighborhood of $\gamma\tau_1$, and let $U_{2,\gamma}$ be a neighborhood of τ_2 disjoint from $U_{1,\gamma}$ (since $\mathcal{H}$ is Hausdorff, such neighborhoods exist). Then

$$U_1 = W_1 \cap \left(\bigcap_{\gamma\in\mathcal{I}'} \gamma^{-1}(U_{1,\gamma}) \right)$$

is a neighborhood of τ_1 and

$$U_2 = W_2 \cap \left(\bigcap_{\gamma\in\mathcal{I}'} U_{2,\gamma} \right)$$

is a neighborhood of τ_2.

We claim that if $\gamma(U_1) \cap U_2 \neq \emptyset$, then $\gamma \notin \mathcal{I}'$. Otherwise, we have $U_1 \subseteq \gamma^{-1}(U_{1,\gamma})$ and $U_2 \subseteq U_{2,\gamma}$ so that $\emptyset \neq \gamma U_1 \cap U_2 \subseteq \gamma^{-1}(U_{1,\gamma}) \cap U_{2,\gamma}$, a contradiction since $U_{1,\gamma}$ and $U_{2,\gamma}$ are chosen to be disjoint. □

Corollary 5 $Y(1)$ *is Hausdorff*

Proof: Let $x_1, x_2 \in Y(1)$ be distinct. Then $x_1 = \pi(\tau_1), x_2 = \pi(\tau_2)$ for some $\tau_1, \tau_2 \in \mathcal{H}$ such that $\gamma\tau_1 \neq \tau_2$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Choosing U_1, U_2 as in the lemma above, $\pi(U_1)$ and $\pi(U_2)$ are disjoint neighborhoods of x_1 and x_2. □

The $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathcal{H}^*$

With an understanding of the quotient topology on $Y(1)$ at hand, we can see that $Y(1)$ is not a compact space - if it were, the fundamental domain $\mathcal{F}$ would also be compact, but as $\mathcal{F}$ is unbounded along the imaginary axis, it is not compact. If, however, we were to add a point at ∞ to $\mathcal{H}$ and extend the fundamental domain $\mathcal{F}$ to include this pint, we could make the image compact. In order to make this compatible with the $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathcal{H}$, we must consider how $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ acts on ∞. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we want γ to act continuously. Since

$$\lim_{\tau \to \infty} \frac{a\tau + b}{c\tau + d} = \frac{a}{c},$$

this requires us to extend the $\mathrm{SL}_2(\mathbb{Z})$ action to $\mathbb{Q} \cup \{\infty\}$. Therefore, we let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$, and define the action on rational numbers as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{r}{t} = \frac{ar + bt}{cr + dt}.$$

We define

$$X(1) = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}^* = Y(1) \cup \mathrm{SL}_2(\mathbb{Z})\infty,$$

and call $\mathrm{SL}_2(\mathbb{Z})\infty$ the cusp at infinity.

In this section, we will show that $X(1)$ is a compact, connected, Hausdorff surface. To do so, we must first define a topology on $\mathcal{H}^*$ such that under the quotient topology on $X(1)$ we will have a connected, compact, Hausdorff space on which we can place a complex structure.

A basis for the topology on $\mathcal{H}^*$ is as follows:

- For $\tau \in \mathcal{H}$, we have the usual discs that lie in $\mathcal{H}$ and are centered at τ.
- For $\tau \in \mathbb{Q}$, we take open discs of $\mathcal{H}$ that are tangent to the real axis at τ.
- For $\tau = \infty$, we have the sets $\mathcal{N}_r = \{\tau \in \mathcal{H} : \mathrm{Im}(\tau) > r\}$, $r > 0$.

We again denote the quotient map $\mathcal{H}^* \to X(1)$ by π, and we define $V \subseteq X(1)$ to be open if and only if $\pi^{-1}(V)$ is open. As π is continuous and $\mathcal{H}^*$ is connected, we again have that $X(1)$ is connected. We claim that $X(1)$ is Hausdorff. Let $x_1, x_2 \in X(1)$ are distinct with $\pi(\tau_1) = \pi(\tau_2)$. If $\tau_1, \tau_2 \in \mathcal{H}$, then we are done by Cor. 5. Suppose next that $\tau_1 \in \mathcal{H}$, $\pi(\tau_2) = \infty$; without loss of generality, we may assume that $\tau_2 = \infty$. Let U_1 be a neighborhood of τ_1 with compact closure K in $\mathcal{H}$, and let

$$R = \max\{\mathrm{Im}(\gamma\tau) : \tau \in K, \gamma \in \mathrm{SL}_2(\mathbb{Z})\}.$$

Then for $U_2 = \{\mathrm{Im}(\tau) > R\} \cup \{\infty\}$, we have $\gamma U_1 \cap U_2 = \emptyset$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, so $\pi(U_1) \cap \pi(U_2)$ are disjoint open sets.

Having shown that $X(1)$ is connected and Hausdorff, it remains to show that $X(1)$ is compact.

Proposition 6 *$X(1)$ is compact*

Proof: Let $\{V_i\}$ be an open cover of $X(1)$. Then $\{\pi^{-1}(V_i)\}$ is an open cover of $\mathcal{H}^*$. There is a set V_0 within the open cover such that $\pi^{-1}(V_0)$ contains ∞. Then the set $\mathcal{F} \setminus \pi^{-1}(V_0)$ is a closed, and bounded set, hence is compact. Since $\{\pi^{-1}(V_i)\}$ covers $\mathcal{F} \setminus \pi^{-1}(V_0)$, there is a finite subcover $\pi^{-1}(V_1), \pi^{-1}(V_2), \ldots, \pi^{-1}(V_n)$ of $\mathcal{F} \setminus \pi^{-1}(V_0)$. Then $V_0, V_1, \ldots, V_n$ covers $X(1)$. $\square$

The complex structure on $X(1)$

In the final section, we'll describe a complex structure on $X(1)$. Having done so, we will have shown that $X(1)$ is a compact Riemann surface. We must identify an open cover $\{V_i\}$ of $X(1)$ and maps ψ_i satisfying the condtions in Def. 1.

First, we let $\mathcal{F}^*$ denote

$$\{\tau \in \mathcal{H} : -1/2 < \mathrm{Re}(\tau) \leq 1/2, |\tau| > 1\} \cup \{\tau \in \mathcal{H} : \mathrm{Re}(\tau) > 0, |\tau| = 1\} \cup \{\infty\}.$$

Claim 7 *The stabilizer of ∞ is $\langle T \rangle$.*

With this claim, recalling Prop 9(b) of lecture 1, we have that the stabilizer of $\tau \in \mathcal{F}^*$ is $\{\pm I_2\}$ if $\tau \notin \{\bar{\omega}, i, \infty\}$, $\langle S \rangle$ if $\tau = i$, $\langle TS \rangle$ if $\tau = -\bar{\omega}$, and $\langle T \rangle$ if $\tau = \infty$.

Let $x \in X(1)$, and let τ_x be the unique element of $\mathcal{F}^*$ such that $\pi(\tau_x) = x$. As shown above, for each such x, we can find a neighborhood U_x of τ_x such that $\gamma U_x \cap U_x = \emptyset$ for all γ such that $\gamma\tau \neq \tau$. In other words, $\gamma U_x \cap U_x = \emptyset$ for $\gamma \notin Stab_{\tau_x}$. The set U_x cover $X(1)$, so if we can find appropriate maps ψ_x, we will have shown that $X(1)$ is a compact Riemann surface. We will first define the maps ψ_x and refer the reader to [4, Thm. 19.9] or [2, Thm. I.2.5] for a proof that this defines a complex structure on $X(1)$.

If $x \in X(1)$ is not the cusp at infinty (which we will also denote by ∞ from now on), then let $\mathbb{D}$ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and let $g_x : \mathcal{H} \to \mathbb{D}$ be defined by

$$g_x(\tau) = \frac{\tau - \tau_x}{\tau - \bar{\tau}_x}.$$

We will define a map from $\pi(U_x)$ to $\mathbb{D}$. When $Stab_{\tau_x} = \{\pm I_2\}$, π restricted to U_x is a homeomorphism so the map $\psi_x = g_x \circ \pi^{-1}$ will be a homeomorphsm from U_x to an open subset of $\mathbb{D}$. When $|Stab_{\tau_x}| = 2n_x$, $n_x > 1$, the restriction of π to U_x is no longer injective. To correct this, we define $\psi_x(z) = g_x(\pi^{-1}(z))^{n_x}$. Finally, we define $g_\infty = e^{2\pi i\tau}$ for $\tau \in \mathcal{H}$, $g_\infty(0) = 0$, and $\psi_\infty = g_\infty \circ \pi^{-1}$.

Theorem 8 *The open cover $\{U_x\}$ with ψ_x described above is an complex structure on $X(1)$,*

Proof: See [4, Thm. 19.9] or [2, Thm. I.2.5]

References

[1] Ahlfors, L., *Complex analysis, third edition*, McGraw Hill, 1979.

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