1 Definitions and Notations

1. Recall that we have an extended upper half plane $H^*$, or $\mathbb{H}^*$, via adding a projective line to $\mathbb{H}$, $H^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Here, we can regard our elements of $\mathbb{H} \subseteq H^*$ as column vectors $\begin{bmatrix} \tau \\ 1 \end{bmatrix}$. We regard elements of $\mathbb{Q} \cup \{\infty\}$ as equivalence classes of column vectors: a rational number $a/b \in \mathbb{Q}$ is regarded as $\begin{bmatrix} a \\ b \end{bmatrix}$, and is equivalent to $\begin{bmatrix} ra \\ rb \end{bmatrix}$ for all $r \in \mathbb{Q} \times \mathbb{Q}$. We also set $\infty := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2. Via matrix multiplication, we have an action of $SL_2(\mathbb{Z})$ on $H^*$ which extends the usual action on $\mathbb{H}$: for $\gamma := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ and $\begin{bmatrix} x \\ y \end{bmatrix} \in H^*$, we set $\gamma \cdot \begin{bmatrix} x \\ y \end{bmatrix} := \frac{ax + by}{cx + dy}$.

3. For any congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, we also have that $\Gamma$ acts on $H^*$. The orbits of $\mathbb{P}(\mathbb{Q})$ under $\Gamma$ are called the cusps of $\Gamma$.

4. Let $X$ and $Y$ be Riemann surfaces and $f : X \to Y$ a nonconstant holomorphic map. Fix $x \in X$, and set $y = f(x)$. If $u$ and $t$ are local parameters$^1$ at $x$ and $y$, respectively, which map $x$ and $y$ to the origin, then in some neighborhood of $x$ we can express $f$ in the form

$$t(f(z)) = a_{e}u(z)^{e} + a_{e+1}u(z)^{e+1} + \cdots, \quad a_{e} \neq 0$$

for some positive integer $e$. This integer is independent of the choice of $u$ and $t$. It is called the ramification index of the covering map $f$ at $x$. If $e > 1$, then $x$ is said to be a ramified point of $f$, and that $y$ ramifies in $X$ under $f$.

The following definitions concern a generalization of modular groups, called Fuchsian groups. They will be used in Problems 5, 6, 10 and 19.

5. A Fuchsian group is a discrete subgroup of $SL_2(\mathbb{R})$. In particular, $SL_2(\mathbb{Z})$ and all of its subgroups are Fuchsian groups.

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$^1$In particular, for some open subsets $U, U' \subset Y$, one has that $u : U \to \mathbb{C}$ and $t : U' \to \mathbb{C}$ are maps which are homeomorphic onto their images, and the transition maps $u \circ t^{-1} : t(U \cap U') \to u(U \cap U')$ and $t \circ u^{-1} : u(U \cap U') \to t(U \cap U')$ are both holomorphic maps.
6. A non-scalar element of $\alpha$ of $GL_2^+(\mathbb{R})$ is called elliptic, parabolic, or hyperbolic when it satisfies

$$\text{tr}(\alpha)^2 < 4 \det(\alpha), \quad \text{tr}(\alpha)^2 = 4 \det(\alpha), \quad \text{or} \quad \text{tr}(\alpha)^2 > 4 \det(\alpha)$$

respectively.

7. A Fuchsian group $\Gamma$ acts on $\mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ via linear fractional transformations. One can show that an element $\alpha \in \Gamma$ is:

- elliptic if and only if $\alpha$ has fixed points $z_0$ and $\overline{z}_0$ for some $z_0 \in \mathcal{H}$;
- parabolic if and only if $\alpha$ has a unique fixed point on $\mathbb{R} \cup \{\infty\}$;
- hyperbolic if and only if $\alpha$ has two distinct fixed points on $\mathbb{R} \cup \{\infty\}$.

8. Fix a Fuchsian group $\Gamma$, and let $z \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$. We call $z$ an elliptic point, parabolic point, or hyperbolic point of $\Gamma$ if there is some elliptic/parabolic/hyperbolic element of $\Gamma$ fixing $z$, respectively.

9. Fix a Fuchsian group $\Gamma$.

- Let $P_\Gamma$ denote the set of parabolic points of $\Gamma$. Elements of $P_\Gamma$ are sometimes called cusps of $\Gamma$.
- The space $\mathcal{H}^*$ denotes $\mathcal{H} \cup P_\Gamma$.
- The space $X(\Gamma)$ denotes the quotient space $\Gamma \backslash \mathcal{H}^*$.

10. Fix a Fuchsian group $\Gamma$, and let $\pi : \mathcal{H}^* \to \Gamma \backslash \mathcal{H}^* = X(\Gamma)$ be the quotient map. A point $a \in X(\Gamma)$ is called an elliptic point or a cusp, respectively, when there is a lift $z \in \mathcal{H}^*$ of $a$ that is either an elliptic point or a cusp for $\Gamma$. When $a$ is neither an elliptic point nor a cusp, it is called an ordinary point.

2 Introductory Problems

Problem 1. Determine the stabilizers of $i, \zeta_3$ and $\infty$ under $SL_2(\mathbb{Z})$, where $\zeta_3 := \frac{-1 + \sqrt{-3}}{2}$.

Problem 2.

a. Prove that $SL_2(\mathbb{Z})$ has exactly one cusp.

b. Show that any congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$ has finitely many cusps.

Problem 3. $SL_2(\mathbb{Z})$ acts on $\mathcal{H}$ properly discontinuously. In other words, for any two points $x, y$ of $\mathcal{H}$, there exist neighborhoods $U$ and $V$ of $x$ and $y$, respectively, such that $\# \{ \gamma \in SL_2(\mathbb{Z}) : \gamma U \cap V \neq \emptyset \} < \infty$. Convince yourself that this is the case.

Problem 4.

1. Show that $\mathbb{C}$ is a Riemann surface.

2. At what points is the map $\mathbb{C} \to \mathbb{C}, z \mapsto z^2$ ramified?

Problem 5. Let $\alpha \in GL_2^+(\mathbb{R})$ be a non-scalar element. Show that the listed definitions for $\alpha$ to be elliptic/parabolic/hyperbolic are indeed equivalent.

Problem 6. Let $\Gamma$ be a Fuchsian group and $\alpha \in \Gamma$ a non-scalar element.

1. Show that if $\alpha$ is an elliptic/parabolic/hyperbolic element of $\Gamma$, then for any $\gamma \in \Gamma, \gamma \alpha \gamma^{-1}$ is also an elliptic/parabolic/hyperbolic element, respectively.

2. How do the fixed points of $\alpha$ and $\gamma \alpha \gamma^{-1}$ compare?

Problem 7. Show that homothetic lattices have equal $j$-invariants.

Problem 8. The $j$-function has a Laurent series expansion in terms of $q := e^{2\pi i \tau}$,

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 86429970q^3 + 20245856256q^4 + 333202640600q^5 + \ldots$$

By the theory of complex multiplication, one has for imaginary quadratic $\tau \in \mathcal{H}$ that $j(\tau)$ is an algebraic integer. Assuming that $j\left(\frac{1+\sqrt{-163}}{2}\right) \in \mathbb{Z}$, use this $j$-function expansion to show that $e^{\pi \sqrt{163}}$ is very close to an integer.
3 Intermediate Problems

Problem 9 (Diamond & Shurman, Exercise 3.1.4). Show that for a prime \( p \in \mathbb{Z}^+ \), \( \Gamma_0(p) \) has exactly two cusps.

Problem 10.

a. Show that every elliptic element of \( \text{SL}_2(\mathbb{Z}) \) is of order dividing 4 or 6.

b. What elements of \( \text{SL}_2(\mathbb{Z}) \) represent the conjugacy classes of elliptic elements?

c. What are the elliptic points of \( \text{SL}_2(\mathbb{Z}) \)?

Problem 11 (Miyake, Lemma 1.7.1). Let \( G \) be a topological group acting continuously on \( X \). Assume that for any two points \( x, y \) of \( X \), there exist neighborhoods \( U \) of \( x \) and \( V \) of \( y \) such that \( gU \cap V = \emptyset \) for all \( g \in G \) satisfying \( x \neq y \). Show that \( G \backslash X \) is a Hausdorff space.

Problem 12.

1. Show that the projective line \( \mathbb{CP}^1 := \mathbb{P}^1(\mathbb{C}) \) is a Riemann surface.

2. There is a map \( \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \) given by \([s : t] \mapsto [s^2 : t^2]\). Where is this map ramified, and what ramification indices does it have at those points?

3. Do the same for the map \( \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \) given by \([s : t] \mapsto [s^2(s-t) : t^3]\).

Problem 13 (Cyclic isogenies). Let \( \mathbb{C}/\Lambda \) be a complex elliptic curve. An isogeny \( \varphi : \mathbb{C}/\Lambda_1 \rightarrow \mathbb{C}/\Lambda_2 \) is called cyclic if its kernel \( \{z + \Lambda_1 \in \mathbb{C}/\Lambda_1 : z \in \Lambda_2\} \) is a cyclic subgroup of \( \mathbb{C}/\Lambda_1 \).

a. Show that a cyclic subgroup \( C \subseteq \mathbb{C}/\Lambda \) induces a cyclic isogeny \( \mathbb{C}/\Lambda \rightarrow \mathbb{C}/C_0 \) with kernel \( C \) for some superlattice\(^2\) \( C_0 \) of \( \Lambda \).

b. Show that any isogeny \( \varphi : \mathbb{C}/\Lambda_1 \rightarrow \mathbb{C}/\Lambda_2 \) factors as a multiplication-by-\( n \) map followed by a cyclic isogeny.

The following four exercises are related to (complex) elliptic curves with complex multiplication, see Problem 11 of Problem Set 3.

Problem 14. Recall that an order \( \mathcal{O} \) of a number field \( K \) is a subring of the ring of integers \( \mathcal{O}_K \) of equal \( \mathbb{Z} \)-rank. Equivalently, \( \mathcal{O} \) is a subring of \( \mathcal{O}_K \) with its own \( \mathbb{Z} \)-basis of algebraic integers. One has that the index \( |\mathcal{O}_K : \mathcal{O}| < \infty \).

a. Show that an order in an imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \) with squarefree \( d \in \mathbb{Z}^+ \) has the form

\[
\mathcal{O} = [1, f \omega_K]
\]

where

\[
\omega_K := \begin{cases} 
\frac{1+\sqrt{-d}}{2} & \text{if } d \equiv 3 \pmod{4} \\
\sqrt{-d} & \text{if } d \equiv 1, 2 \pmod{4}
\end{cases}
\]

and \( f = [\mathcal{O}_K : \mathcal{O}] \).

b. Show that for each integer \( f \in \mathbb{Z}^+ \), the lattice \( \mathcal{O}_f := [1, f \omega_K] \) is an order of \( K \) with index \( f \) in \( \mathcal{O}_K \).

Problem 15. Let \( \mathbb{C}/\Lambda \) be a complex elliptic curve with CM. Then its endomorphism ring \( \mathcal{O} := \text{End}(\mathbb{C}/\Lambda) \) is an order in an imaginary quadratic number field \( K \).

For an endomorphism \( \alpha \in \text{End}(\mathbb{C}/\Lambda) \), we write \( (\mathbb{C}/\Lambda)[\alpha] \) for its kernel \( \ker \phi_\alpha = \alpha^{-1}\Lambda/\Lambda \). We call this the \( \alpha \)-torsion subgroup of \( \mathbb{C}/\Lambda \).

Let us assume the following fact: as \( \mathcal{O} \)-modules, we have for \( \alpha \in \text{End}(\mathbb{C}/\Lambda) \) that

\[
(\mathbb{C}/\Lambda)[\alpha] \cong_{\mathcal{O}} \mathcal{O}/\alpha \mathcal{O}.
\]

Then show that the degree of an endomorphism \( \alpha \in \text{End}(\mathbb{C}/\Lambda) \) is the absolute value of its field-theoretic norm, \( \deg(\phi_\alpha) = |\text{Nm}_{K/Q}(\alpha)| \).

\(^{2}\)A superlattice of \( \Lambda \) is less fun than it sounds: it is just a lattice which contains \( \Lambda \). Compare this word to sublattice.
Problem 16. Show that for two isogenous complex elliptic curves \( \mathbb{C}/\Lambda_1 \) and \( \mathbb{C}/\Lambda_2 \), \( \mathbb{C}/\Lambda_1 \) has CM iff \( \mathbb{C}/\Lambda_2 \) has CM.

Problem 17.

a. Show that if a lattice \( \Lambda \subseteq \mathbb{C} \) is homothetic to its complex conjugate \( \overline{\Lambda} \), then \( j(\Lambda) \in \mathbb{R} \). (In fact, this is if and only if.)

b. Show that if \( \mathcal{O} \) is an order in an imaginary quadratic number field, then \( j(\mathcal{O}) \in \mathbb{R} \).

c. Conclude that for any imaginary quadratic order \( \mathcal{O} \), there is some complex elliptic curve \( \mathbb{C}/\Lambda \) with CM by \( \mathcal{O} \) and whose \( j \)-invariant is a real number. (Hint: assume that Problem 11.c on Problem Set 3 works if we replace \( \mathcal{O}_K \) with \( \mathcal{O} \).)

4 Advanced Problems

Problem 18. Let \( f : X \to Y \) be a nonconstant holomorphic map of compact Riemann surfaces.

a. Show that \( f \) is surjective.

b. Show that \( f \) has finite fibers: that is, for all \( y \in Y \) one has \( \# f^{-1}(y) < \infty \).

Note that there is analogous statement in algebraic geometry: any nonconstant morphism \( \phi : C_1 \to C_2 \) of projective algebraic curves is surjective and has finite fibers.

Problem 19. This problem will construct a Fuchsian group which has no cusps.

a. Consider the following real 2 by 2 matrices:

\[
\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}, \quad \gamma = \frac{1}{2}(1 + \alpha + \beta + \alpha \beta).
\]

(In the definition of \( \gamma \), the element 1 is being used to denote the identity matrix.) Show that \( \mathcal{O} = \mathbb{Z} \oplus \mathbb{Z} \alpha \oplus \mathbb{Z} \beta \oplus \mathbb{Z} \gamma \) is a (noncommutative but unital) subring of the ring \( M_2(\mathbb{R}) \) of 2 by 2 real matrices. This is a quaternion algebra, as the elements \( \alpha \) and \( \beta \) satisfy \( \alpha^2 = -1, \quad \beta^2 = 3, \quad \alpha \beta = -\beta \alpha \).

b. Consider the conjugation on \( \mathcal{O} \) given for any element of the form \( a = a_0 + a_1 \alpha + a_2 \beta + a_3 \alpha \beta \in \mathcal{O} \) by

\[
\bar{a} = a_0 - a_1 \alpha - a_2 \beta - a_3 \alpha \beta.
\]

Show that conjugation \( a \mapsto \bar{a} \) defines a ring automorphism of \( \mathcal{O} \).

c. For any \( a \in \mathcal{O} \), show that \( a + \bar{a} = \text{tr}(a), \quad a\bar{a} = \text{det}(a) \).

Here, \( \text{tr} \) and \( \text{det} \) are the usual trace and determinant operations on matrices.

d. Let \( \mathcal{O}_1 = \{ a \in \mathcal{O} : a\bar{a} = 1 \} \).

Show that \( \mathcal{O}_1 \) is a Fuchsian group with no cusps. (Hint: Write \( a \) as in part b., and write down the condition that \( a \) would satisfy if it were parabolic explicitly in \( a_0, a_1, a_2, a_3 \). Then do a “descent” procedure by looking modulo 3.)

One can use this problem to show that \( \mathcal{O}_1 \setminus \mathcal{H} \) is a compact Riemann surface.