Quadratic Forms and the local-global principle

Lecture 6: Introduction to the theta correspondence

Recall from Prof Barrios:

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \tau} = 1 + 2e^{2\pi i} + 2e^{8\pi i} + \cdots$$

$$= 1 + 2q + 2q^4 + 2q^9 + \cdots$$

$$q = e^{2\pi i \tau}$$

For $k \in \mathbb{Z}_{\geq 1}$:

$$\theta^k(\tau) = \sum_{n \geq 0} r_k(n) q^n, \text{ where}$$

$r_k(n) =$ # of ways to write n as a sum of k squares

$= \#\{(x_1, \ldots, x_k) \in \mathbb{Z}^k : x_1^2 + \cdots + x_k^2 = n\}$

$\underline{k=4}$. • θ^4 is a modular form!
(Lecture 6 of Barrios)

• Wk4 (HW4, #11) Every positive # can be written as a sum of 4 squares. $\Rightarrow$ Every Fourier coeff in θ^4 is $\neq 0$

This is telling us:

counting solns to quad forms $\rightsquigarrow$ Fourier coeffs of modular forms.

This is the beginning of a very long story...

Today. "toy model" for theta corr

Part 1: Orthogonal groups $/\mathbb{F}_q$

$\rightsquigarrow$ theta correspondence

Part 2: Weil rep $/\mathbb{F}_q$

Ex. θ^4

Fourier transform $/\mathbb{F}_q$

Part 3: adeles

Part I: Orthogonal groups / $\mathbb{F}_q$

Let (V, Q) be any quad. space / $\mathbb{F}_\ell$.

Def. The orthog gp corr to V is:

$$O(V) := \{ g \in GL(V) : h_Q(vg, wg) = h_Q(v, w) \ \forall v, w \in V \}$$

Ex. 2dim'l quad spaces:

$$V_{sp} = (\mathbb{F}_q^{\oplus 2}, \mathrm{Hyp}) \quad , \quad V_{ns} = (\mathbb{F}_{\ell^2}, \mathrm{Nm})$$

can show:

$$O(V_{sp}) \cong \mathbb{F}_q^\times \rtimes \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\text{"det"}} \qquad (SO(V_{sp}) := SO(V_{sp}) \cap SL_2 \| \mathbb{F}_\ell^\times)$$

$$O(V_{ns}) \cong \ker(\mathbb{F}_{\ell^2}^\times \to \mathbb{F}_\ell^\times) \rtimes \mathbb{Z}/2\mathbb{Z}$$

Def. $\mathrm{Fun}(V) := \{ \varphi : V \to \mathbb{C} \}$.

↑ is a vector space over $\mathbb{C}$

$$\dim = \# V = q^{\dim V}$$

We know: $V \circlearrowleft O(V)$

$\rightsquigarrow \mathrm{Fun}\,V \circlearrowleft O(V)$

$$\rho: O(V) \times \mathrm{Fun}\,V \longrightarrow \mathrm{Fun}\,V$$
$$(g, \varphi) \longmapsto (v \mapsto \varphi(vg^{-1})).$$

This defines an action of $O(V)$ on $\mathrm{Fun}(V)$

Moreover, this action is linear!

Remarkable: there is another group which also acts on $\mathrm{Fun}(V)$ in a way that doesn't interfere w/ the $O(V)$ action.

$$SL_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a,b,c,d \in \mathbb{F}_q \\ ad-bc=1 \end{array} \right\}$$

mysterious!
nontrivial!
involves Fourier transform!

Part 2. The finite-field Weil representation

Prof. Watson's comm:

$SL_2\mathbb{Z}$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$SL_2\mathbb{F}_q$ is NOT !

Prop. $SL_2\mathbb{F}_q$ is generated by

$$d(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad u(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$a \in \mathbb{F}_q^\times$ $\qquad$ $b \in \mathbb{F}_q$

Moreover, every elt g of $SL_2\mathbb{F}_q$ can be written uniquely in one of the following ways:

- $g = d(a) \cdot u(b)$ for some $a \in \mathbb{F}_q^\times$, $b \in \mathbb{F}_q$
- $g = d(a) \cdot u(b_1) \cdot s \cdot u(b_2)$ for some $a \in \mathbb{F}_q^\times$, $b_1, b_2 \in \mathbb{F}_q$.

Fix: $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ fixed nontriv char

Def. The Fourier transform of a fn $\varphi : V \to \mathbb{C}$ is: $\varphi \in \mathrm{Fun}(V)$

$$FT(\varphi) : V \to \mathbb{C}, \quad v \mapsto \frac{1}{q^{\dim V/2}} \sum_{w \in V} \varphi(w) \cdot \psi(-h_Q(v,w))$$

dim 1: $\frac{1}{\sqrt{q}}$ $\qquad$ $-xy$

Def. Def $\omega : SL_2\mathbb{F}_q \times \mathrm{Fun}(V) \longrightarrow \mathrm{Fun}(V)$ by

- $\omega(u(b), \varphi)(v) := \psi\left(\frac{b}{2} Q(v)\right)\varphi(v)$ $\quad b \in \mathbb{F}_q$
- $\to$ $\omega(d(a), \varphi)(v) := z_a \varphi(av)$ $\quad a \in \mathbb{F}_q^\times$
- $\underline{\omega(s, \varphi)(v)} := z_s \mathrm{FT}(\varphi)(v)$

for some roots of unity z_a, z_s. $\in \mathbb{C}^1$.
$a \in \mathbb{F}_q^\times$

<u>Thm.</u> (Weil rep) $\exists$ choice of z_a, z_s s.t.

ω defines a linear action of $SL_2\mathbb{F}_q$ on $\mathrm{Fun}(V)$.

Rmk: $\omega(s, \omega(s, \varphi))(v) = z_s^2 \mathrm{FT}(\mathrm{FT}(\varphi))(v)$

$s^2 \Leftrightarrow \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ $\quad \Downarrow$ — Fourier inv.

$z_s^2 \varphi(-v)$

$$\omega(s^2, \varphi)(v) = \omega\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \varphi\right) = \omega(d(-1), \varphi)$$

$$= z_{-1}\varphi(-v)$$

Thm $\Rightarrow$ $z_s^2 \varphi(-v) = z_{-1}\varphi(-v) \Rightarrow z_s^2 = z_{-1}$.

Lemma. The $O(V)$-actn on $\mathrm{Fun}(V)$ commutes w/ the $SL_2\mathbb{F}_q$-action on $\mathrm{Fun}(V)$.

Pf. Want: $\underline{g \in O(V)}$, $h \in SL_2\mathbb{F}_q$

$$\rho(g, \omega(h,\varphi)) = \omega(h, \rho(g,\varphi))$$

Recall: $\rho(g,\varphi)(v) = \varphi(vg)$.

Let's check $\rho(g, \omega(s,\varphi)) = \omega(s, \rho(g,\varphi))$

$$\rho(g, \omega(s,\varphi))(v) = \omega(s,\varphi)(vg)$$

$$= \frac{z_s}{q^{\dim V/2}} \sum_{w \in V} \varphi(w)\psi(-h_Q(vg, w))$$

$$-h_Q(vg,w) = -h_Q(v, wg^{-1})$$

$$= \frac{z_s}{q^{\dim V/2}} \sum_{w \in V} \varphi(wg)\psi(-h_Q(v,w))$$

$$= \frac{z_s}{q^{\dim V/2}} \sum_{w \in V} \rho(g,\varphi)(w)\psi(-h_Q(v,w))$$

$$= z_s \mathrm{FT}(\rho(g,\varphi))(v)$$

$$= \omega(s, \rho(g,\varphi))(v). \quad \square$$

What have we done:

Start: quad space V

$\downarrow$

$$\underline{SL_2\mathbb{F}_q} \overset{\omega}{\circlearrowright} \mathrm{Fun}(V) \overset{\rho}{\circlearrowleft} \underline{O(V)}$$

$\mathbb{C}$-vectorspace

can do this for $\mathbb{Q}_p$ replacing $\mathbb{F}_q$!!

(after handling some technical issues)

& $\mathbb{R}$!!

$\downarrow$

build this picture for

$$\mathbb{A}_{\mathbb{Q}} \subset \mathbb{R} \times \prod_{p<\infty} \mathbb{Q}_p$$

$\rightsquigarrow$ mod. forms & aut. forms.

Part 3. The adeles.

Def. The ring of adeles over $\mathbb{Q}$ is

$$\mathbb{A}_{\mathbb{Q}} := \left\{ (x_\infty, x_2, x_3, x_5, \dots) \in \mathbb{R} \times \prod_{p<\infty} \mathbb{Q}_p : \; x_p \in \mathbb{Z}_p \text{ for all but f. many } p<\infty \right\}.$$

mult & add are componentwise.

Exercise: $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$
$x \longmapsto x_p$

Show that $\mathbb{Q}$ embeds diagonally in $\mathbb{A}_{\mathbb{Q}}$.

Analogously, can define

$SL_2(\mathbb{A}_{\mathbb{Q}}) :=$ restr. product $SL_2\mathbb{R} \times \prod_{p<\infty} SL_2\mathbb{Q}_p$ w.r.t. $SL_2\mathbb{Z}_p$ for $p<\infty$.

There is a dictonary $\phi_f : SL_2(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$

modular form $f: \mathfrak{h} \to \mathbb{C}$	$\rightsquigarrow$ automorphic forms. ϕ_f
f, fn on $\mathfrak{h}$	behavior of ϕ_f on $SL_2\mathbb{R}$
wt of f	transformatn of ϕ_f under translati by $S^1 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right.$ $\overset{\parallel}{\mathbb{C}^1}$
level N	invariance of ϕ_f under congruence subgps of $SL_2\mathbb{Z}_p$ at $p \mid N$.
$SL_2\mathbb{Z}$	$SL_2\mathbb{Z}_p\ \forall p$