Week 1

Arizona Winter School 2021:
Quadratic Forms and the Local-Global Principle

Quadratic spaces.

<u>Def.</u> V = f. dim'l v-sp over k. A <u>quadratic form</u> on V is fn $Q: V \to k$ s.t.

(1) $Q(ax) = a^2 \cdot Q(x) \quad \forall a \in k, x \in V$

(2) the fn $V \times V \to k$

$$(x,y) \mapsto Q(x+y) - Q(x) - Q(y)$$

is a bilinear form.

A morphism of quad forms $(V,Q) \to (V',Q')$ is a v.sp hom $\varphi: V \to V'$ s.t. $Q' \circ \varphi = Q$.
If φ is an isom, often say φ is an <u>isometry</u>.

<u>Observe</u>: ◦ fn in (2) is symm

◦ $(x,x) \mapsto Q(2x) - Q(x) - Q(x) = 4Q(x) - 2Q(x)$
$= 2Q(x)$.

Assume forever that char $k \neq 2$.

Set $h_Q(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right)$.
$h_Q: V \times V \to k$

Have: $\{\text{quad forms } V \to k\} \longleftrightarrow \{\text{symm bilin form } V \times V \to k\}$
$Q \mapsto h_Q$

Ex: (a) $V = k \oplus k$

$Q: V \to k, \quad (x, y) \mapsto xy$

$h_Q(v_1, v_2) = \frac{1}{2}\left(Q(v_1+v_2) - Q(v_1) - Q(v_2)\right)$

$v_1 = (x_1, y_1)$
$v_2 = (x_2, y_2)$

$= \frac{1}{2}\left((x_1+x_2)(y_1+y_2) - x_1y_1 - x_2y_2\right)$

$= \frac{1}{2}(x_1y_2 + x_2y_1).$

(b) $V =$ Quad field extn of k.

$Q: V \to k, \quad x \mapsto Nm(x)$

More explicitly: Pick $d \in k$, squarefree, Set $V = k[\sqrt{d}]$

$Q(x+y\sqrt{d}) = (x+y\sqrt{d})(x-y\sqrt{d}) = x^2 - dy^2.$

What is h_Q?

Pick a basis $e_1, \ldots, e_n$ of V.

Define $\underbrace{A = (a_{ij})_{i,j=1,\ldots,n}}_{\text{symmetric matrix}}$ by $a_{ij} := h_Q(e_i, e_j)$

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$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad h_Q(e_j, e_i) = a_{ji}$

change basis by $X \in GL_n(k)$, then

A gets replaced by XAX^t.

Obs: $\det(A)$ $\qquad\qquad \det(A) \cdot \underline{\det(X)^2}$

$\det(A)$ dep on the choice of basis, but only up to an elt of $(k^*)^2$.

$\therefore$ can define

$$\mathrm{disc}(Q) := \text{im of } \det(A) \text{ in } k^*/(k^*)^2.$$

Ex: Write down an A for Ex (a), (b)

What is the discriminant in these cases?

Orthogonality.

(V, Q) quad space + choice of basis of V $\rightsquigarrow$ A symm matrix

This video: $\exists$ a choice of basis s.t. A is diagonal.

Fix (V, Q) quad space.

- $x, y \in V$ are orthogonal if $h_Q(x, y) = 0$.
- for any subset $S \subset V$, let
$$S^\perp := \{ v \in V : h_Q(v, s) = 0 \ \ \forall s \in S \}$$
- $V_1, V_2 \subset V$ subspaces. We say V_1, V_2 are orthogonal if $V_1 \subset V_2^\perp$
- $V^\perp$ = orthogonal complement of V itself
  = the radical of V
- Q is nondegenerate if $V^\perp = 0$.

- $x \in V$ is <u>isotropic</u> if $Q(x) = 0$
- $x \in V$ is <u>anisotropic</u> if $Q(x) \neq 0$
- a quad sp is <u>anisotropic</u> if every nonzero vector is anisotropic.

<u>Lemma</u>. If (V, Q) is nondegen, then

$$V \longrightarrow \mathrm{Hom}(V, k)$$
$$v \longmapsto (w \mapsto h_Q(v, w))$$

is an isomorphism.

<u>Pf</u>. If $h_Q(v,w) = 0 \ \forall w \in V$,

then $v \in V^\perp = 0 \Rightarrow v = 0 \Rightarrow$ injectivity.

$\dim V = \dim \mathrm{Hom}(V, k) \Rightarrow$ also get surj. $\square$

<u>Prop</u>. If $U \hookrightarrow V$ is s.t. $Q|_U$ is nondegenerate,

then $V = U \oplus U^\perp$.

<u>Pf</u>. Nondegen of $U \Rightarrow U \cap U^\perp = 0$.

ETS: $V = U + U^\perp$

Take $v \in V$ & consider the lin fnl $\begin{array}{l} U \to k \\ w \mapsto h_Q(w, v). \end{array}$

By Lemma, $\exists u \in U$ s.t. $h_Q(w,v) = h_Q(w,u) \ \forall w \in U$.

$\Rightarrow h_Q(w, v-u) = 0 \ \forall w \in U \Rightarrow v - u \in U^\perp$. $\square$

Thm. Every quadratic space has an orthog basis.

Pf: Induct on dim of V.

- If $V^\perp = V$, then any basis of V is an orthogonal basis of V.

- If $V^\perp \subsetneq V$, then $\exists\, e_1 \in V$ anisotropic.

$$\not\approx 0$$

$\Rightarrow ke_1$ is a nondegen. quad sp.

so we can apply Prop $U = ke_1$.

$$\Rightarrow V = ke_1 \oplus \underbrace{ke_1^\perp}_{\dim n-1}$$

$\square$

Let $e_1, \ldots, e_n$ be an orthog basis of V.

Then the assoc. matr $A = (a_{ij})$

$$a_{ij} = h_Q(e_i, e_j) = 0 \quad \text{if } i \neq j$$

$\Rightarrow A$ is diag

Moreover: rank A = # nonzero elts along diag

= codim of $V^\perp$

Ex: Find an orthog basis for Ex(a), (b).

Zero spaces, hyperbolic spaces, and anisotropic spaces

In this video:

every quadratic sp is a sum of:

- zero space (radical $V^\perp$)
- split space (hyperbolic space)
- nonsplit space (anisotropic part)

Note: orthogonality $\Rightarrow$ can always split off the radical

(V, Q) is a fixed nondegen quad sp.

Def. A hyperbolic plane is a two-dim'l quadratic space (H_2, Q) s.t. $\exists$ a basis v_1, v_2 of H_2 satisfying:

$$Q(v_1) = Q(v_2) = 0, \quad h_Q(v_1, v_2) = 1.$$

A $\underbrace{\text{hyperbolic space of dim } 2r}_{H_{2r}} \cong (H_2)^{\oplus r}$

Ex. (a) is a hyperbolic plane!

(b) is not! anisotropic

Thm. $V \cong H_{2r} \oplus W$ where W is anisotropic.

Moreover, such a decomp is unique up to isom.

Witt's theorem — see notes & problem set.

Pf of existence. Repeated application of the following key proposition:

Key prop. If V contains a nonzero isotropic vector, then V contains a hyperbolic plane.

Pf. $x \in V$ nonzero isotropic vector; i.e. $Q(x) = 0$.

nondeg of $Q \Rightarrow \exists\, y \in V$ s.t. $h_Q(x,y) = 1$.

Claim: $H := \mathrm{span}\{x, y\}$ is a hyperbolic plane.

Pf. Take $v_1 := x$

$v_2 := y + \lambda x$

$$\left\{\begin{array}{l} h_Q(v_1, v_2) = 1 \\ Q(v_1) = 0 \\ \text{remain: } Q(v_2) = 0 \end{array}\right.$$

$$\begin{aligned} Q(v_2) &= Q(y + \lambda x) \\ &= h_Q(y+\lambda x, y+\lambda x) \\ &= Q(y) + \lambda\big(h_Q(x,y) + h_Q(y,x)\big) \\ &= Q(y) + 2\lambda \Rightarrow \lambda = \frac{-Q(y)}{2} \text{ then } Q(v_2) = 0 \end{aligned}$$

$\square$

Universality:

- A nondegen quad fm (V, Q) is *universal* if
$$Q(V) \supset k^*.$$

<u>Cor.</u> Any nondegen quad space with a nonzero isotropic vector is universal.

<u>Pf.</u> A hyperbolic plane is universal. $\square$

<u>Q:</u>
- Why do we need nondegeneracy in the cor?
- Does every universal quad space contain a nonzero isotropic vector?

Hint: Ex (b).