2.1 Absolute values

We'll begin this lecture by formalizing many of our thoughts from last lecture, in particular, how we think of notions of "size."

Definition 2.1

For a field k, an **absolute value** on k is a function

$$|\cdot| : k \to \mathbb{R}_{\geq 0}$$

such that

i) $|x| = 0$ if and only if $x = 0$

ii) $|xy| = |x||y|$ for all $x, y \in k$

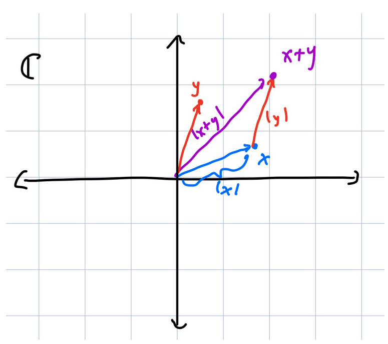
iii) $|x + y| \leq |x| + |y|$, the "triangle inequality"

We say that an absolute value $|\cdot|$ is **nonarchimedean** if in addition

iv) $|x + y| \leq \max\{|x|, |y|\}$

Note that iv implies iii.

Example 0: $k = \mathbb{C}$. For $z \in \mathbb{C}$ (with $a = \mathrm{Re}(z), b = \mathrm{Im}(z)$), the magnitude function $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$ is an absolute value. Here is the picture of the triangle inequality for this example:



Example 1: $k = \mathbb{Q}$. We will denote the "usual" absolute value by $|\cdot|_\infty$, defined as:

$$|x|_\infty = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

2.1.1 Example p: the p-adic absolute value

Again let $k = \mathbb{Q}$. Let p be a prime. Last lecture, we thought of numbers highly divisible by p as "small." We measure divisibility by p via a valuation

$$v_p : \mathbb{Z} - \{0\} \to \mathbb{R}$$

defined as follows. For each $n \in \mathbb{Z} - \{0\}$, let $v_p(n)$ be the unique integer satisfying

$$n = p^{v_p(n)} n' \text{ with } p \nmid n'$$

For example, since

$$200 = 5^2 \cdot 8 = 3^0 \cdot 200 = 2^3 \cdot 25$$

we have

$$v_5(200) = 2, \;\; v_3(200) = 0, \;\; v_2(200) = 3$$

We extend v_p to $\mathbb{Q}^\times$ via

$$v_p\left(\frac{a}{b}\right) := v_p(a) - v_p(b)$$

and we can also set $v_p(0) := +\infty$.

You should check that this is well-defined and does not depend on the representative a/b of a rational number. You can also check the following convenient way of calculating valuations: if $x = p^k \frac{a'}{b'}$ and a', b' are integers not divisible by p, then $v_p(x) = k$.

For example, since

$$3/200 = 5^{-2} \cdot \frac{3}{8} = 3^1 \cdot \frac{1}{200} = 2^{-3} \cdot \frac{3}{25}$$

we have

$$v_5(3/200) = -2, \;\; v_3(3/200) = 1, \;\; v_2(3/200) = -3$$

Theorem 2.2

This valuation satisfies the properties

1. $v_p(xy) = v_p(x) + v_p(y)$
2. $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$

Proof sketch: 1. We show this for $x \in \mathbb{N}_{>0}$. We write $x = p^{v_p(x)} n_x$ and $y = p^{v_p(y)} n_y$ with $p \nmid n_x$, $p \nmid n_y$. Then $xy = p^{v_p(x)+v_p(y)} n_x n_y$, and $p \nmid n_x n_y$ since p is prime. This relies on the primality of p and would be false for, say, the 10-adics.
2. WLOG, $v_p(x) \geq v_p(y)$. So

$$x + y = p^{v_p(y)} (p^{v_p(x) - v_p(y)} n_x + n_y).$$

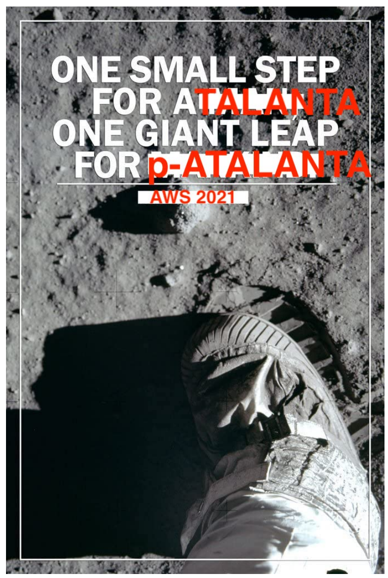
The green quantity is an integer, so $v_p(x+y) \geq v_p(y)$. $\square$

We bring this together to make things highly divisible by p *small* in an absolute value that also records the number p; for $x \in \mathbb{Q}^\times$ we set

$$|x|_p := p^{-v_p(x)} \text{ and } |0|_p = 0$$

Examples: $|\frac{100}{3}|_3 = 3^1 = 3, |\frac{100}{3}|_5 = 5^{-2} = \frac{1}{25}$. We also note that in p-atalanta's course, the steps were of 5-adic size $|4 \cdot 5^i|_5 = \frac{1}{5^i}$.

Atalanta, on the other hand, stepped in intervals of $1/2^i$. And $1/2^10$ for example is very small with respect to the $|\cdot|_\infty$ absolute value, but when $p-2$, the p-adic absolute value is $|1/2^10|_2 = 1024$. So $1/2^10$ is...



Theorem 2.3

$|\cdot|_p$ is a non-archimedean absolute value.

Proof sketch: $p^{-\text{Theorem 2.1.1}}$.

2.2 Nonarchimedean Strangeness

Let's explore the concept of nonarchimedean a little more, since it's really a new type of metric.

Firstly

Theorem 2.4 "All triangles are isoceles"

For a nonarchimedean metric $|\cdot|$, if $|x| \neq |y|$, then $|x+y| = \max\{|x|, |y|\}$

Proof: WLOG, suppose $|x| > |y|$. Then

$$|x+y| \leq |x| = |x+y-y| \leq \max\{|x+y|, |y|\}$$

so $|x| = |x+y|$ □

This makes the absolute value of a sum easy to calculate if the summands have different absolute values. For example, $|4 \cdot 5 + 25k|_5 = |4 \cdot 5|_5 = 1/5$.

It also has some strange consequences. Suppose that p-atalanta (with $p = 5$) starts at 0, and strides forth one meter at a time. After 2 strides, she is at distance $|2|_5 = 1$ from 0. Even more surprising, after 5 strides, she is at distance 1/5 from 0, closer than her first step! We can generalize this phenomenon into a characterization of nonarchimedean metrics:

Theorem 2.5

Let k be a field, and let ϕ be the unique nonzero ring homomorphism $\phi : \mathbb{Z} \to k$, which sends n to $\sum_{i=1}^{n} 1$. Then an absolute value $|\cdot|$ is nonarchimedean if and only if $|\phi(n)| \leq 1$ for all $n \in \mathbb{Z}$.

Note: we will abuse notation by writing n in lieu of $\phi(n)$ in k.
Proof: $\Rightarrow$: we induct on n. From the nonarchimedean property, $|n \pm 1| \leq \max\{|n|, 1\} \leq 1$.

$\Leftarrow$: let $x, y \in k$. We want to show that $|x+y| \leq \max\{|x|, |y|\}$. This is trivial if $|y| = 0$, so further assume that $y \neq 0$. Then, by multiplicativity, what we want to show is equivalent to showing that $|x/y+1| \leq \max\{|x/y|, 1\}$, so we need only show that $|x+1| \leq \max\{|x|, 1\}$.

To show this, we take powers and use the binomial coefficent theorem. For $n \geq 2$, we consider

$$|x+1|^n = \left|\sum_{i=0}^{n} \binom{n}{i} x^i\right| \tag{2.1}$$

$$\leq \sum_{i=0}^{n} \left|\binom{n}{i} x^i\right| \tag{2.2}$$

$$\leq \sum_{i=0}^{n} |x|^i \tag{2.3}$$

$$\leq (n+1)\max\{1, |x|^n\} \tag{2.4}$$

where line 2.2 follows from the triangle inequality for absolute values, line 2.3 follows from the assumption on the absoute value of integers, and line 2.4 follows from the fact that $|x|^n > 1$ if and only if $|x| > 1$.

Taking nth roots,

$$|x+1| \leq (n+1)^{1/n} \max\{1, |x|\}.$$

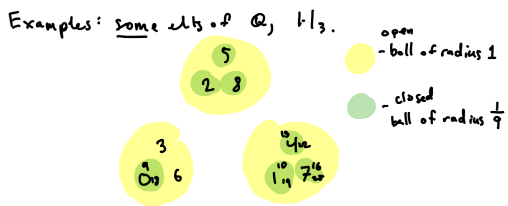
Taking the limit as n goes to $+\infty$ yields the result.

Conversely, for archimedean absolute values, integers can be arbitrarily large. Another characterization of archimedean absolute values is that for any nonzero $s \in k$, and any real number B, there exists $n \in \mathbb{N}$ such that $|ns| > B$. That is, for a bathtub with any volume B of water, and a spoon which can hold volume s, you can eventually empty the bathtub spoonful by spoonful, as depicted in the figure below provided by Joanne Beckford:

2.3 Having a Ball

Definition 2.6

1. For $x, y \in k$ and an absolute value $|\cdot|$ on k, we define the **distance between** x **and** y as $d(x, y) = |x - y|$
2. Let $r > 0$. The **open ball centered at** x **with radius** r is $B(x, r) := \{z \in k : |z - x| < r\}$
3. Let $r > 0$. The **closed ball centered at** x **with radius** r is $B_{cl}(x, r) := \{z \in k : |z - x| \leq r\}$



Nonarchimedean balls are very different from archimedean balls!

Theorem 2.7

1. Every point of a ball is "the center" of the ball
2. Every open ball is closed
3. Every closed ball is open
4. Two balls can only intersect if one is contained within the other.

Proof: exercise.

2.4 Common Values

This topology looks very different from what we are used to in our physical world. We may ask ourselves if these absolute values are just strange outliers.

Definition 2.8

We say that two absolute values $|\cdot|$ and $|\cdot|'$ are **equivalent** if and only if there exists $\alpha \in \mathbb{R}^+$ such that for all $x \in k$,

$$|x|' = |x|^\alpha$$

We could also define absolute values as being equivalent if they induce the same topology. The equivalence of these definitions is an exercise.

Theorem 2.9 Ostrowski's Theorem

Every nontrivial absolute value $|\cdot|$ on $\mathbb{Q}$ is either equivalent to $|\cdot|_p$ for some prime p (if $|\cdot|$ is nonarchimedean), or $|\cdot|_\infty$ (if $|\cdot|$ is archimedean).

Proof: First, suppose that $|\cdot|$ is archimedean. Let n_0 be the least element of $\mathbb{N}$ such that $|n_0| > 1$, and let $\alpha \in \mathbb{R}$ such that $|n_0| =^\alpha_0$. We'll show that $|n| = n^\alpha$ for all $n \in \mathbb{N}$, and the result for $\mathbb{Q}$ follows by multiplicativity.
We write

$$n = b_0 + b_1 n_0 + b_2 n_0^2 + ... + b_k n_0^k$$

for some $b_i \in \{0, 1, ..., n-1\}$ and $k \in \mathbb{N}$. So

$$|n| = |b_0 + b_1 n_0 + b_2 n_0^2 + ... + b_k n_0^k| \tag{2.5}$$

$$\leq |b_0| + |b_1| n_0^\alpha + |b_2| n_0^{2\alpha} + ... + |b_k| n_0^{k\alpha} \tag{2.6}$$

$$\leq 1 + n_0^\alpha + n_0^{2\alpha} + ... + n_0^{k\alpha} \tag{2.7}$$

$$\leq n_0^{k\alpha} \frac{n_0^\alpha}{n_0^\alpha - 1} \tag{2.8}$$

by the minimality of n_0 and comparison with the geometric series. Let $C := \frac{n_0^\alpha}{n_0^\alpha - 1}$, noting that it is constant and does not depend on k. Then for all $N \in \mathbb{N}$, $n^N \leq C n^{N\alpha}$, so taking Nth roots and taking limits, $|n| \leq n^\alpha$.
The other inequality is similar: bound terms and take limit.
Next, suppose that $|\cdot|$ is nonarchimedean. By Theorem 2.2, $|n| \leq 1$ for all $n \in \mathbb{Z}$. Let k be the least nonzero natural number such that $|k| < 1$. If k were not prime, so if there were a nontrivial factorization in $\mathbb{Z}$ as $k = ab$, we would have $|a||b| < 1$, so WLOG $|a| < 1$, a contradiction. We will now denote by p the least nonzero natural number with absolute value less than 1.
Let $n \in \mathbb{N}$ such that $p \nmid n$. Then, by the division algorithm, $\exists k \in \mathbb{Z} : n = kp + r$ with $0 < r < p$. Since $r < p$, by the definition of p, $|r| = 1$. So by the "all triangles are isoceles" proposition,

$$|n| = |kp + r| = \max\{|kp|, |r|\} = 1$$

Then if n is an arbitrary nonzero natural number, it can be written as $p^k n'$ with $p \nmid n'$. By multiplicativity, $|n| = |p|^k$ and the result follows. $\square$

Not only do we know all the absolute values on $\mathbb{Q}$, but they also fit together beautifully in the following formula!

Theorem 2.10 Product formula

For any $x \in \mathbb{Q}^\times$, we have

$$|x|_\infty \prod_{p \text{ prime}} |x|_p = 1$$

2.5 Completing Our Discussion

You may have seen the construction of the real numbers as Cauchy sequences of rational numbers up to some equivalence. The sequences were Cauchy with respect to the absolute value $|\cdot|_\infty$; we will now go through this process with respect to the absolute values $|\cdot|_p$. We recall some definitions:

Definition 2.11 A

sequence of elements x_n is called a **Cauchy sequence** if for every $\epsilon \in \mathbb{R}^+$ there exists $M \in \mathbb{N}$ such that for all $m, n > M$, $|x_n - x_m| < \epsilon$.

Examples: for $|\cdot|_\infty$, we said that $3, 3.1, 3.14, 3.141, 3.1415, ...$ was a Cauchy sequence. We can also see that the sequence we constructed earlier for p-atalanta, $x_n := \sum_{i=0}^{n} 4 \cdot 5^i$, for $m > n$ we have that

$$x_m - x_n = \sum_{i=n+1}^{m} 4 \cdot 5^i$$

which has valuation less than $1/5^n$. Hence, (x_n) is a Cauchy sequence with respect to $|\cdot|_5$!

Definition 2.12

A field k is **complete** with respect to an absolute value $|\cdot|$ if every Cauchy sequence has a limit in k.

We note that $\mathbb{Q}$ is not complete with respect to $||_\infty$, since you can find a sequence (x_n) such that $\lim_{n\to\infty} x_n^2 = 2$, but there is no square root of 2 in $\mathbb{Q}$.

Similarly, in the last lecture, we found a sequence of natural numbers $(b_i)_{i\in\mathbb{N}}$ with $b_i \in$

$\{0, 1, ..., 6\}$ such that for

$$x_n := \sum_{i=0}^{n} b_i 7^i$$

we had $7^{n+1} \mid (x_n^2 - 2)$. Hence, $|x_n^2 - 2|_7 \leq \frac{1}{7^{n+1}}$ so $\lim_{n\to\infty} x_n^2 = 2$, but again, $x^2 = 2$ has no solution in $\mathbb{Q}$.

Suppose we want to "add in" the "limits" of these sequences. This is likely how you met $\mathbb{R}$ in your analysis class; it was constructed as the completion of $\mathbb{Q}$ with respect to $||_\infty$.

Definition 2.13

$\mathbb{R} := \{(x_n)_{n\in nn} : x_n \in \mathbb{Q}, (x_n)$ Cauchy with respect to $|\cdot|_\infty\}/\sim$ where $(x_n) \sim (x_n')$ iff $\lim_{n\to\infty} |x_n - x_n'|_\infty = 0$.

We now give a new definition of $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

Definition 2.14

$\mathbb{Q}_p := \{(x_n)_{n\in nn} : x_n \in \mathbb{Q}, (x_n)$ Cauchy with respect to $|\cdot|_p\}/\sim$ where $(x_n) \sim (x_n')$ iff $\lim_{n\to\infty} |x_n - x_n'|_p = 0$.

2.6 Old $\mathbb{Q}_p$ vs New $\mathbb{Q}_p$?

We'll begin discussing the relationship of the two objects we've defined as $\mathbb{Q}_p$ with an analysis of Cauchy series in nonarchimedean metrics.

Theorem 2.15

If $|\cdot|$ is a nonarchimedean metric on a field k, a sequence (x_n) is Cauchy if and only if

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0$$

Proof: if $m > n$ are elements of $\mathbb{N}$, write $m = n + r$. Then by the nonarchimedean triangle inequality,

$$\begin{aligned} |x_m - x_n| &= |x_{n+r} - x_{n+r-1} + x_{n+r-1} - x_{n+r-2} + ... + x_{n+1} - x_n| \\ &\leq \max\{|x_{n+r} - x_{n+r-1}|, ..., |x_{n+1} - x_n|\} \end{aligned}$$

□

We note that this is not true for archimedean absolute values, since, for example, the sequence for the harmonic series $x_n = \sum_{i=1}^{n} 1/i$ does not converge. In fact, series converges for nonarchimedean absolute values is quite nice in this sense!

Theorem 2.16

Let $|\cdot|$ is a nonarchimedean metric on a field k. Let $(s_i)_{i\in\mathbb{N}}$ be a sequence of elements of k, and let

$$\sigma_n = \sum_{i=0}^{n} s_i$$

be a sequence of partial sums. If $\lim_{n\to\infty} |s_n| = 0$, then $(\sigma_n)_n$ is a Cauchy sequence.

Proof: we note that $|\sigma_{n+1} - \sigma_n| = |s_n|$ and apply Theorem 2.6. □

Let's now connect these concepts with some definitions and sequences we saw last lecture. Recall:

Definition 2.17

A sequence of integers a_n such that $0 \le a_n \le p^n - 1$ is **coherent** if for all $n \ge 1$,

$$a_n \equiv a_{n+1} \pmod{p^n}$$

This is a number theoretic condition, but we can see that such a sequence also satisfies an analytic condition!

Theorem 2.18

Let p be a prime. If $(a_n)_{n\in\mathbb{N}}$ is a coherent sequence of integers, then $(a_n)_{n\in\mathbb{N}}$ is Cauchy with respect to $|\cdot|_p$.

Proof: $p^n \mid (a_{n+1}-a_n)$, and so $|a_{n+1}-a_n|_p \le 1/p^n$, so $\lim_{n\to\infty} |a_{n+1}-a_n|_p = 0$. Theorem 2.6 then gives the result. □

Lastly, we will see that the series we defined last lecture are actually limits of Cauchy sequences, so they are elements of the completion!

Theorem 2.19

Let $n_0 \in \mathbb{Z}$ and $(b_i)_{i=n_0}^{\infty}$ be a sequence with $b_i \in \{0, ..., p-1\}$ for all i. Let

$$a_n = \sum_{i=n_0}^{n} b_i p^i.$$

Then $(a_n)_{n\in\mathbb{N}}$ is Cauchy with respect to $|\cdot|_p$.

Proof: $|a_{n+1} - a_n|_p = |b_{n+1}p^{n+1}|_p = 1/p^{n+1}$. Theorem 2.6 then gives the result. □

Next time, we'll view our new $\mathbb{Q}_p$ from yet another angle and complete the connection with our first definition.