Remark on non-abelian reciprocity

11 March, 2020

(Warning: some numbers below may be incorrect because I'm writing much from memory.)

Dear Students,

Many thanks for working so hard this week and the nice presentation. On the drive to the airport, I realised I should have expressed the last point a bit more carefully, because it really is the most important source of interesting problems in the theory. This concerns the interpretation of equations defining global points inside the p-adic points as *explicit non-abelian (or non-linear) reciprocity laws.*

The abelian version is for the variety $\mathbb{G}_m$ and concerns characters of the Galois group G_F of a number field F. If

$$\phi : G_F \longrightarrow \mathbb{Q}_p$$

is such a character, then it factors through $\phi : G_F^{ab} \longrightarrow \mathbb{Q}_p$, so that we can consider the function

$$\phi \circ rec : \mathbb{A}_F^\times = \mathbb{G}_m(\mathbb{A}_F) \longrightarrow \mathbb{Q}_p.$$

Of course, the Artin reciprocity law says that

$$\phi \circ rec((x_v)) = 0$$

when an idele (x_v) comes from $\mathbb{G}_m(F)$. Taking such an equation apart, you will typically see discrete contributions from $v \nmid p$, and analytic contributions from $v \mid p$. (This last should be imposed as a condition on the character). This is because any continuous character $\mathcal{O}_{F_v}^\times \longrightarrow \mathbb{Q}_p$ for $v \nmid p$ is trivial, so that the only contribution will be from a uniformiser π_v. To make this more definite, you can consider the situation where ϕ is unramified outside some set S of places including all places dividing p, and we restrict the reciprocity map to $S-$integral ideles $\mathbb{A}_S^\times$, that is, (x_v) such that x_v is a unit for all $v \notin S$. In that case, the equation $\phi(rec((x_v))) = 0$ will only have contributions from $v \in S$, and will be a union of equations of the form

$$\sum_{v|p} \phi \circ rec_v = \sum_{v \in S, v \nmid p} c_v,$$

where the c_v run over possible discrete contributions from $v \nmid p$.

For a very explicit example, you can consider the case where $F = \mathbb{Q}$ and $\phi = \log \chi_{cyc}$, where

$$\chi_{cyc} : G_\mathbb{Q} \longrightarrow \mathbb{Z}_p^\times$$

is the cyclotomic character. In that case, the equation

$$\phi \circ rec((x_v)) = 0$$

becomes

$$\log(x_p) = \sum_{q \neq p} v_q(x_q) \log q,$$

where the logarithm is the p-adic one. The S-integral version is

$$\log(x_p) = \sum_{q \neq S \setminus p} v_q(x_q) \log q,$$

This can be viewed as an infinite union of zero sets

$$\log(x_p) = \sum_{q \neq S \setminus p} n_q \log q$$

as the n_q run over integers. Obviously, we don't expect any kind of finitieness since, for example, the set of S-units is infinite for S non-empty. On the other hand, if S is empty, we get $\log(x_p) = 0$ for $x_p \in \mathbb{Z}_p$ which is only satisfied by the roots of unity in $\mathbb{Z}_p$. Thus we get close to $\mathbb{G}_m(\mathbb{Z}) = \pm 1$. Note that you get functions that vanish on global points from any function on G_F^{ab} that vanishes at the origin, even if it's not a character.

The first non-abelian reciptocity law I know of was discovered by Coleman, and says

$$D_2(2) = D_2(-1) = D_2(1/2) = 0,$$

where

$$D_2(z) := \ell_2(z) + (1/2)\log(z)\log(1-z),$$

a p-adic 'Rogers dilogarithm', and $\ell_2(z)$ is the dilogarithm given by a power series

$$\ell_2(z) = \sum_{n=1}^{\infty} z^n/n^2$$

for $|z| < 1$ and analytically continued via Coleman integration

$$-\int_{(d/dt)_0}^{z} (dt/t)(dt/(1-t)).$$

Coleman deduced this vanishing from functional equations, but it can also be given a global proof using the fact that $\{2, -1, 1/2\}$ are exactly the $\mathbb{Z}[1/2]$ points of $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. That is, the level-two $\mathbb{Q}_p$-unipotent π_1 in this case is an extension

$$0 \longrightarrow \mathbb{Q}_p(2) \longrightarrow U_2 \longrightarrow \mathbb{Q}_p(1) \times \mathbb{Q}_p(1) \longrightarrow 0.$$

The fundamental diagram looks like

$$\begin{array}{ccccc} X(\mathbb{Z}[1/2]) & \longrightarrow & X(\mathbb{Z}_p) & & \\ \downarrow & & \downarrow & \searrow^{j_{DR}} & \\ H^1_f(G_S, U_2) & \xrightarrow{loc} & H^1_f(G_p, U_2) & \longrightarrow & U_2^{DR} \end{array}$$

(For $\mathbb{P}^1 \setminus \{0, 1\infty\}$, we have $F^0 = 0$.) We are taking $S = \{2, p\}$ with $p \neq 2$, so that

$$H^1_f(G_S, U_2) \simeq H^1_f(G_S, U_1) \simeq H^1_f(G_S, \mathbb{Q}_p(1) \times \mathbb{Q}_p(1)) \simeq \mathbb{A}^2,$$

while U_2^{DR} has dimension 3. It turns out in this case that with respect to suitable coordinates A, B, C, the image of loc is defined by $2C - AB = 0$, while the map j_{DR} is given by

$$j_{DR}(z) = (\log(z), -\log(1-z), \ell_2(z)),$$

Thus, Coleman's equations follow from the 2-integrality.

Subsequent to this, many further reciprocity laws have been found. For example, the integral points on a punctured elliptic curve

$$E \setminus O : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

of rank 1 with semistable reduction everywhere lies in the set

$$\cup_w \{ \int_b^z \frac{dx}{(2y + a_1x + a_3)} \frac{xdx}{(2y + a_1x + a_3)} + [\frac{a_1^2 + 4a_2 - Eis_2(E, \alpha)}{12} - \frac{h(y)}{\log^2(y)}] \log^2(z) = \|w\| \},$$

where w runs over a finite set of $|S|$-tuples of numbers $(w_v)_{v \in S}$, S being the primes of bad reduction, and $\|w\| = \sum_v |w_v|$. The precise statement is in the paper 'A non-abelian conjecture of Tate-Shafarevich type'. The w that occur here are determined by the bad fibres. The number $Eis_2(E, \alpha)$ is the value of a p-adic Eisenstein series at the pair $(E, dx/(2y + a_1x + a_3))$. Finally, y is a Heegner point and $h(y)$ its p-adic height. In numerical examples, it's quite surprising to see the quantity actually vanish on integral points, since the double integral and

$$\log(z) = \int_b^z \frac{dx}{(2y + a_1x + a_3)}$$

are algebraically independent as functions on $E(\mathbb{C}_p)$. The earliest correct version of this was found after much agonising only with the help of Jennifer and Kiran.

The equations you have found for the $2,3$-integral points on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ are of course further reciprocity laws of this nature. You will have noticed that they are quite hard to compute. Perhaps this shouldn't be surprising since any such equation is after all of the same order of complexity as Artin's reciprocity law, which only deals with the simple variety $\mathbb{G}_m$.

In general, given a refined Selmer variety Sel_n^Σ of level n, where the superscript Σ refers to conditions imposed at primes of bad reduction, any element of the defining ideal of

$$\overline{loc(\mathrm{Sel}_n^\Sigma)} \subset H^1_f(G_p, U_n)$$

will give you a reciprocity law. The challenge is to find such elements for any given curve X. Going further, one might hope for *canonical* elements lying in the ideals that give you especially pretty (and useful) reciprocity laws.

Best,

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