2020 ARIZONA WINTER SCHOOL PROJECT NOTES: "ABELIAN CHABAUTY"

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1. Suggestions for preparation

For almost all of our proposed projects, it is essential to have a working knowledge of Chabauty's method. In addition to the "Abelian Chabuaty" course notes, we recommend starting with McCallum and Poonen's survey [MP12].

For the more computational projects, we recommend Poonen's surveys [Poo96] and [Poo02]. More importantly, it is important to quickly come up wtih speed with how to perform Chabauty's method in Magma. Magma has a free, limited use online calculator here

`http://magma.maths.usyd.edu.au/calc/`,

and a thoroughly documented implementation of Chabauty's method

`http://magma.maths.usyd.edu.au/magma/handbook/text/1533`.

Even better is to obtain a copy for your laptop, or ssh access to a departmental server with a copy of Magma. The Simons Foundation has graciously made Magma freely available to mathematicians working in the US

`http://magma.maths.usyd.edu.au/magma/ordering/`;

please contact your department's tech staff, who should be able to help you obtain a copy of Magma through this agreement.

Finally a very useful exercise is to take Smart's list of the 427 genus 2 curves with good reduction away from 2, and provably find all of the rational points on them. I have set up a temporary folder at my web page

`http://www.math.emory.edu/~dzb/AWS2020`

containing several references, and containing a subfolder titled "preparatory-Magma-exercise" with instructions for this exercise.

2. Project Descriptions

Project 1. Quadratic points on modular curves. The goal is of this project is to determine the K-rational points on certain modular curves using a modification of Chabauty's method, where K is a quadratic number field. More precisely, there is a heuristic of Siksek and Wetherell [Sik13] saying that for a nice curve $X/\mathbb{Q}$, a Chabauty-type method could bound the number of K-rational points on a curve X of genus g under the weaker assumption that $J_X(K)$ has rank $r \le d(g-1)$ where $d = [K : \mathbb{Q}]$.

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The works of [BN15, OS19] focused on determining the quadratic points on the modular curves $X_0(N)$ of genus ≤ 5 with Mordell–Weil rank 0, and the work of [Box19] studied the cases when the Mordell–Weil rank is positive. The modular curve $X_0(37)$ stands out because it has genus 2, positive Mordell–Weil rank, and two sources of infinitely many quadratic points: one coming from the hyperelliptic map $X_0(37) \to \mathbb{P}^1$ and one coming from the quotient by the Atkin–Lehner involution $X_0(37) \to X_0(37)^+$ [Box19, Section 5].

While the project heading is quite broad, it would be interesting to start with a study of the quadratic points on $X_0(37)$ and to see when the heuristic of Siksek and Wetherell can be applied. More precisely:

(1) For $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{-3})$, can one provably determine $X_0(37)(K)$ using the heuristic of Siksek and Wetherell? Are there other quadratic fields K (perhaps real quadratic field) where $X_0(37)(K)$ can be provably determined?
(2) For the above mentioned fields K, can one determine where the quadratic points come from using the geometry of $X_0(37)$?
(3) For $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(\sqrt{-3})$, can one provably determine $X_0(43)(K)$ using the heuristic of Siksek and Wetherell?

Recommended reading. A discussion of the heuristic of Siksek and Wetherell can be found at [Sik13], and for a detailed example of the heuristic of Siksek and Wetherell, we recommend reading [Doy18, Appendix A]. We also recommend the works [BN15, OS19, Box19] for thorough discussions of quadratic points on modular curves.

Project 2. Rank functions for special families of curves. The goal of this project is to improve the "rank favorable" bounds on the rank functions that arise in [Sto06] and [KZB13] for special curves (e.g., trigonal). This project would involve very little p-adic analysis; the techniques are more akin to the geometry of curves and combinatorics.

Recommended reading. We recommend reading [Sto06] and [KZB13] to get an understanding for the rank functions involved in their work.

Project 3. Rank favorable bounds for special families of curves. In [Sto13], Stoll proved uniform bounds on rational points of hyperelliptic curves over $\mathbb{Q}$ with low Mordell–Weil, where the bounds incorporate the Mordell–Weil rank of the curve (i.e., the lower the Mordell–Weil rank of the hyperelliptic curve is the better the bound becomes). A key ingredient in getting rank favorable uniform bounds is that one has an explicit description of the differentials on a hyperelliptic curve which helps with the "p-adic analysis" part of the arguments. Moreover, one can hope to find rank favorable uniform bounds for special families of curves where one has an explicit description of differentials. In [Kan17], Kantor accomplished this by determining rank favorable uniform bounds for superelliptic curves.

The goal of this project is to determine rank favorable uniform bounds for "special" families of curves, where one has an explicit description of differentials. Here is a guideline:

(1) Determine uniform bounds for non-hyperelliptic genus 3 curves (i.e., plane quartics) of Mordell–Weil rank 0.
(2) Determine uniform bounds for plane curves of low Mordell–Weil rank.

The main theorem of [KRZB15] gives uniform bounds in both of these cases, but we anticipate that one can obtain much better bounds in both cases.

Recommended reading. We recommend [KRZB18] for a survey of the techniques involved in determining uniform bounds for curves of low Mordell–Weil rank. Also, we recommend the original works of Stoll [Sto13] and Katz–Rabinoff–Zureick-Brown [KRZB15].

Project 4. Uniform bounds for the dth symmetric products of curves with small rank. The goal of this project is to combine works on symmetric power Chabauty and on uniform bounds for curves of low Mordell–Weil rank to obtain uniform bounds for the dth symmetric products of curves with low Mordell–Weil rank. In [VW17], Vemulapalli–Wang determined uniform bounds for symmetric squares of curves of low Mordell–Weil rank, which also satisfy another technical assumption (cf. [GM17, Assumption 5.7(†)]).

Once participants have an understanding of the tools involved in symmetric power Chabauty and in the works on uniform bounds for curves of low Mordell–Weil rank (e.g., p-adic analysis, non-Archimedean geometry, and some tropical geometry), the project boils down to a problem in combinatorics. Here is a guideline:

(1) Determine how small the Mordell–Weil rank of a curve needs to be in order to incorporate the symmetric power Chabauty and uniform bound techniques.
(2) Find uniform bounds for the dth symmetric product of curves with the above rank condition.

Recommended reading. We recommend [Sik09] for the foundations of symmetric power Chabauty and [GM17] for how to use tropical techniques to make symmetric power Chabauty explicit. Also, the work [VW17] illustrates the combinatorial nature of the project.

Project 5. Avoiding dth symmetric powers in a $d+1$st symmetric power. It would be interesting, but possibly substantial, to improve the work of Siksek and Box to the case where $\mathrm{Sym}^d X(\mathbb{Q})$ is infinite, and to find the points of $\mathrm{Sym}^{d+1} X(\mathbb{Q})$ which are not in the image of $X(\mathbb{Q}) \times \mathrm{Sym}^d X(\mathbb{Q}) \to \mathrm{Sym}^{d+1} X(\mathbb{Q})$.

As an example: there are several composite, but non prime power, level modular curves that arise in naturally in "Mazur's program B" for which the determination of rational points has some particular challenging aspect. The example that inspired this is the following. The modular curve $X_0(65)$ is genus 5, and not trigonal; there are finitely many cubic points on $X_0(65)$. However, $\mathrm{Sym}^3 X_0(65)(\mathbb{Q})$ is not finite. The Jacobian $J_0(65)$ decomposes as $E \times A_1 \times A_2$, where E is a rank 1 elliptic curve, and both A_i are geometrically simple rank 0 abelian surfaces; moreover, the optimal map $X_0(65) \to E$ has degree 2. Since $X_0(65)$ has rational points (the 4 cusps), there are 4 "copies" of $\mathrm{Sym}^2 X_0(65)(\mathbb{Q})$ lying on $\mathrm{Sym}^3 X_0(65)(\mathbb{Q})$. In particular, $\mathrm{Sym}^3 X_0(65)(\mathbb{Q})$ is infinite, but only finitely many rational points of $\mathrm{Sym}^3 X_0(65)$ do not lie on one of the copies of $\mathrm{Sym}^2 X_0(65)(\mathbb{Q})$.

Recommended reading. Suppose that X is a curve and $f\colon X \to C$ is a degree d map. This gives rise to a map $C \to \mathrm{Sym}^d X$. When $C(\mathbb{Q})$ is infinite (e.g., C is $\mathbb{P}^1$ or a positive rank elliptic curve), this gives rise to infinitely many degree d points on X.

The papers [Sik09] and [Box19] explain how to determine the degree d points of X which do not arise from C. We are hopeful that a modification of their argument will work.

Project 6. Improved rank favorable bounds. This project is recommend for anyone who is looking for a purely combinatorial project.

The "rank favorable bounds" part of my lecture notes discusses the following theorem.

Theorem 2.1 (Stoll [Sto06]; Katz, Zureick-Brown, [KZB13])**.** *Let $X/\mathbb{Q}$ be a curve of genus g and let $r = \operatorname{rank} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2r + 2$ is a prime, that $r < g$, and let $\mathscr{X}$ be a proper regular model of X over $\mathbb{Z}_p$. Then*

$$\#X(\mathbb{Q}) \leq \#\mathscr{X}_{\mathbb{F}_p}^{sm}(\mathbb{F}_p) + 2r.$$

Actually, Stoll observed that if X is not hyperelliptic, then one can further improve the $2r$ term in the bound. Define

$$f_X(r) = \max\left\{\deg(D) \,|\, D \geq 0 \text{ and } \dim H^0(X, \Omega^1(-D)) \geq g - r\right\};$$

then $f(r) \leq 2r$ (for $0 \leq r < g$), with equality if an only if X is hyperelliptic, and the bound in Stoll's theorem is actually

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + f(r).$$

See [Sto06, Section 3].

The bound in the bad reduction case can be similarly improved, if one instead defines

$$f_\Gamma(r) = \max\{\deg(D) \,|\, D \geq 0, \text{and} \dim r(K - D) \geq g - r - 1\}$$

for a graph Γ with canonical divisor K. Here, one can work with either the "numerial rank" (i.e., the notion of rank from [Bak08, 1.3]), or the "abelian rank" (from [KZB13, 3.3]).

Problem: Understand $f(r)$ for non-hyperelliptic graphs. (See [BN09] for the notion of hyperelliptic graph.) For example:

- We know that if $f(r) = 2r$, then Γ is hyperelliptic. Contrapositively, if Γ is not hyperelliptic, then $f(r) < 2r$. Can we characterize graphs such that $f(r) < 2r - 1$?
- Find interesting families for which $f(r)$ is small. (For example: what happens for trigonal graphs, or for graphs with extra automorphisms?)

Project 7. A curve with many rational points. The "Elkies–Stoll" curve

$$\begin{aligned} X\colon y^2 = &82342800x^6 - 470135160x^5 + 52485681x^4 + \\ &2396040466x^3 + 567207969x^2 - 985905640x + 247747600 \end{aligned}$$

has at least 642 rational points, and its Jacobian has rank 22. See

`http://www.mathe2.uni-bayreuth.de/stoll/recordcurve.html`

for a full list of the known points.

It would be interesting to prove that $\#X(\mathbb{Q})$ is **exactly** 642. Reducing mod a few primes of good reduciton verifies that $\operatorname{Jac}_X(\mathbb{Q})_{\text{tors}}$ is trivial. Magma's RankBound command computes that the rank is at most 22, and differences of known points generate a subgroup of rank 22; [MS16, Proposition 19.1] proves that the subgroup generated by the known points is the full group, and exhibits explicit generators.

Problem: Prove that $\#X(\mathbb{Q}) = 642$.

Chabauty's method is clearly not applicable (since $22 \geq 2$). One untested idea is to pass to a field extension where X attains a 2-torsion point and to attempt a combination of étale descent and elliptic Chabauty. (Of course, there are a few things that need to go right, and it is possible that this approach won't work; but, if it doesn't work, it would be useful to know that! There is a longer interesting list of curves at [Sto09, Figure 5] for which it would be interesting to provably compute $X(\mathbb{Q})$.)

Update, added March 2. The field one would need to work over for this problem has degree 15, which is probably too large to do computations with, even assuming GRH (though I would prefer to be proven wrong!). There are a few simpler, but still interesting, curves with many points that are easier to study. In the document

`http://www.math.harvard.edu/~elkies/many_pts.pdf`

Elkies disucsses several examples. The record before the Elkies–Stoll curve above was due to Keller-Kulesz; the curve

$$X_2\colon y^2 = 278271081x^2(x^2-9)^2 - 229833600(x^2-1)^2$$

has 588 rational points, and it obtains a 2-torsion point over a quadratic extension. Moreover, its Jacobian is isogenous to the square of an elliptic curve of rank at least 12. It would be interesting to determine $X_2(\mathbb{Q})$ explicitly.

See also [Poo96, Section 8] for a discussion of other interesting examples.

Recommended reading. Poonen's surveys [Poo96] and [Poo02] are a good start. The paper [RZB15] has several examples of similar (but easier) computations (for example Subsections 8.3 and 9.2; see also the accompanying transcript of computations [RZB]).

Poonen's notes "Lectures on rational points on curves", available at

`http://www-math.mit.edu/~poonen/papers/curves.pdf`

are also a great resource.; see for example Section 7 on étale descent.

Project 8. Point Count Records. The uniformity conjecture is the following.

Conjecture 2.2 ([CHM97])**.** Let K be a number field and let $g \geq 2$ be an integer. There exists a constant $B_g(K)$ such that for every smooth curve X over K of genus g, the number $\#X(K)$ of K-rational points is at most $B_g(K)$.

This famously follows from Lang's conjecture, and inspired a large effort to compute lower bounds on the constants $B_g(K)$. It would be interesting to take another look at the literature, starting with [Poo96, Section 8],

`http://www.math.harvard.edu/~elkies/many_pts.pdf`,

[Sto09] (and possibly some of the papers discussed there, such as [Kul98, Kul95, KK95]), and to see if the methods there can be improved (especially for fields $K \neq \mathbb{Q}$). For example: it would be really interesting to generate examples of genus 2 curves over $\mathbb{Q}$ with a very large number of quadratic points; such a curve would likely have very large rank (and as of now, we have better records for ranks of elliptic curves (28) than ranks of simple abelian surfaces (22, I think)).

Project ∞. Rational points on (symmetric powers of) modular curves. Find every rational point on every (symmetric power of every) modular curve. More seriously: there are several interesting example of modular (in some appropriate sense) curves, and one collection of projects is to study, via Chabauty and other explicit methods, the rational points on these curves.

Recommended reading. We recommend taking a look at [RZB15] and [Poo02].

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